Maturity structure of debt and self-fulfilling crises*

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Abstract

This paper investigates the role of debt maturity within a model of self-fulfilling debt crises. Using one-period bonds, we characterize an interval of debt levels where creditors' panic can force a government to default. We numerically show that the government optimally lowers debt to reduce its vulnerability to this type of crises. We then switch to a model with long-term debt repayable with an infinite stream of coupons. In this environment, we show that the equilibrium price of debt differs from the short-term debt scenario. Furthermore, we prove that the bounds of the interval where crisis are self-fulfilling shift upward with higher debt maturity. Finally, we numerically show that the government decreases debt levels faster in the presence of long-term debt, therefore rising the economy's welfare compared to the short-term debt case.

Keywords: Debt Crisis, Sovereign Debt Default, Bond duration, Maturity choice, Maturity structure

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1 Introduction

Recent empirical evidence from the last European debt crisis indicates that sudden changes in creditors' beliefs about governments' repayment capacity can influence the likelihood of defaults. During the crisis, the European Central Bank announced a liquidity support program (Outright Monetary Transactions) to address the rise in bond yields volatility that was disconnected from changes in economic fundamentals. After the announcement, a large decline in yields occurred even though the liquidity support was not used – see Figure 1. This evidence suggests that the promise of liquidity itself could have brought calm to the markets, thus showing a link between the likelihood of default and creditors' beliefs about governments' debt repayment capacity. Using a self-fulfilling debt crises model, several studies have explained this link with a focus on governments' repayment capacity of short-term debt (Cole and Kehoe, 1996, 2000; Conesa and Kehoe, 2017). However, the study of this link with a focus on governments' repayment capacity of long-term debt has largely remained unexplored.

In this paper, we study the impact of long-term debt in a model of self-fulfilling debt crises. The main goal is to analyze how the maturity structure of debt affects governments' default decisions in a model of self-fulfilling debt crisis, and to understand its welfare implications. We show that holding the ratio of debt-over-GDP constant, governments' incentives to default decrease as the maturity of debt increases. This is because a higher debt maturity allows a government to roll over a lower fraction of debt in each period, thus making the government less exposed to negative shifts in expectations. We also show that this decrease in governments' default incentives coming from long-term debt can increase the welfare of the economy. Specifically, a government can decrease total debt to levels that are not vulnerable to creditors' panic, in a shorter period of time compared to the short-term debt case.

The paper consists of two parts. In the first part, we build on an earlier work of Cole and Kehoe (2000) and consider a model of self-fulfilling debt crises featuring short-term debt in a small open economy. A government which maximizes households' welfare chooses spending on a consumption good, issues one-period bonds, and has access to a competitive international credit market. In every period, given the bonds' price, the government first chooses how much debt to sell, and then decides how much to spend and whether to default or not. Upon default, the government is permanently excluded from the bond market, and seizes a share of the economy's endowment thereafter. In this set-up, we first show that there can be an interval of debt levels in which repayment or default are rational expectations equilibria (the *Crisis region*). Within this interval, a sunspot can suddenly trigger creditors' panic and force the government to default, thus confirming creditors' negative initial expectations.

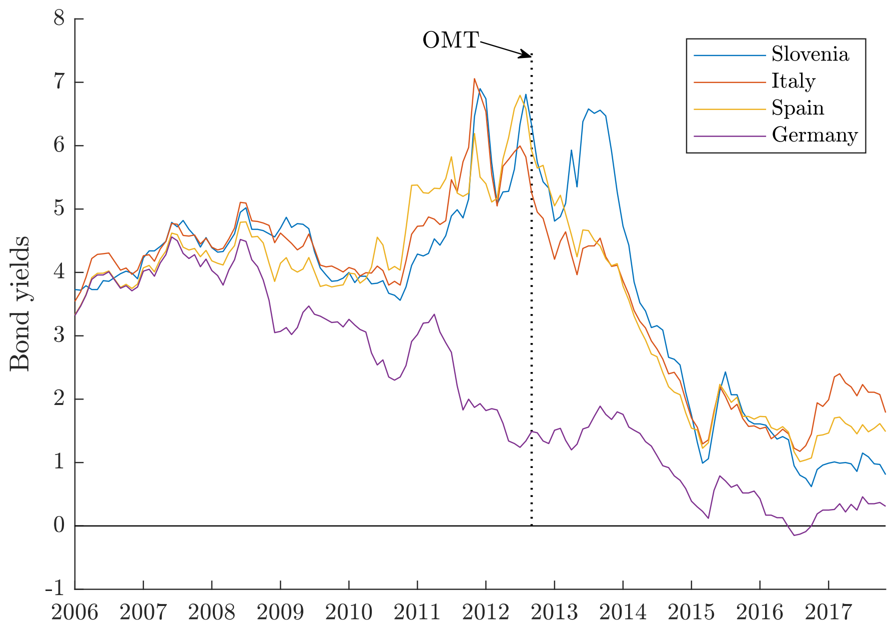


Fig. 1. Outright Monetary Transactions (OMT) announcement and long-term government bond yields (10-year). *Source: OECD. Units: Perc., Not SA. Freq: Month.*

When initial debt lies in the Crisis region, we analytically prove that the government tries to avoid self-fulfilling debt crises by lowering its debt to reach an interval of debt levels where government repayment is the only rational expectations equilibrium (the *No-Default region*). We further show that the decrease in debt levels occurs in a fixed number of periods that optimally depend on how large initial debt is within the Crisis region.

In the second part of the paper, we consider a model of self-fulfilling debt crises using debt of higher maturity in the form of coupon-paying bonds. This definition of long-term debt in line with Hatchondo and Martinez (2009) assumes that creditors receive an infinite stream of coupon payments that geometrically decrease with time. The key aspect of long-term debt in the form of a coupon-paying bond is that it lowers the amount of debt maturing in each period, and therefore allocates the repayment of total debt more evenly across time. Also, this definition of long-term debt explicitly introduces credit market freezes. In this set-up, we first study how the structure of the equilibrium price of debt changes compared to the short-term debt scenario, and analyze the change in the bounds of the Crisis region under long-term debt. In particular, we show that these bounds increase as debt maturity rises, shifting upward the Crisis region relative to the one-period bond case. The intuition is that coupon payments divide the repayment of total debt in many periods, and since government incentives to default on less maturing debt decrease, creditors now expect default to occur only at higher levels of total debt. Second, we numerically investigate the implications of long-term debt in this model and compare it to the short-term debt scenario. Specifically, we describe how the presence of long-term debt shortens the time in which the government decreases any initial debt starting in the Crisis region. Finally, we numerically show that total utility in this model can be higher when the government issues long-term bonds. Intuitively, the fastest the government exits the Crisis region, the less it discounts the payoff attained in the No-Default region. These findings, therefore, suggest that lengthening debt maturity can improve an economy's welfare.

Related literature. This paper contributes to the literature on self-fulfilling debt crises. Two seminal papers of Cole and Kehoe (1996, 2000) explore governments' incentives to repudiate debt when default choices take place after borrowing. They characterize an interval of debt levels (the Crisis region) where creditor's panic can force the government to default, even with unchanged economic fundamentals. The authors analytically show that the government can optimally reduce debt levels in order to avoid this type of crises. This framework with coordination failures has been explored by several other works in the literature to explore options that governments have to escape Crisis region. For example, Conesa and Kehoe (2017) extend Cole and Kehoe's environment allowing a government to 'gamble for redemption', namely, to wait for an economic recovery while keeping large debt levels in the Crisis region. Roch and Uhlig (2018) analyze the impact of a bailout agency in the probabilities of sovereign default. More recently, Szkup (2022) uses global games to explore how austerity and fiscal stimulus affect the probability of self-fulfilling debt crises, and Aguiar et al. (2022) quantitatively study how governments can prevent default by auctioning bonds at low prices. In line with these models, this paper contributes to the literature of self-fulfilling debt crises by analyzing the impact that the management of debt maturity can have on welfare.

This paper relates to the extensive literature on endogenous default started by Eaton and Gersovitz (1981). For example, Aguiar and Gopinath (2006) and Arellano (2008) explore the link between an economy's macroeconomic variables and the probability of sovereign default. Hatchondo and Martinez (2009) and Arellano and Ramanarayanan (2012) introduce different debt maturities to interact them with a government's default choice. In these papers, crises happen as a result of poor economic fundamentals. On the contrary, in this paper we adopt the self-fulfilling debt crises approach from Cole and Kehoe (2000) to analyze the role of debt maturity.

This paper also relates to the sovereign default literature that explores different specifications of debt maturity. Niepelt (2014) uses long-term debt with a bond that matures in two periods. In contrast, Hatchondo and Martinez (2009) model long-term debt with a bond that pays coupons that decay at a fixed rate over time. This bond specification renders a tractable state-space, and is the definition of debt that we will use in our paper. Other works also implemented a similar bond structure. Arellano and Ramanarayanan (2012) combine the bond specification from Hatchondo and Martinez (2009) with a one-period bond. Chatterjee and Eyigungor (2012) define a bond that either matures or pays a coupon with some fixed probability in each period. In more recent works, Aguiar et al. (2022) and Roch and Uhlig (2018) also feature long-term bonds with infinite coupon payments.

The paper proceeds as follows. Section 2 outlines the set-up of the model under short-term debt. Section 3 characterizes a Crisis region and analyzes an optimal debt policy to exit it. The model with long-term debt is introduced in Section 4. Section 5 presents the Crisis region and illustrates government optimal decisions under this new type of debt. Section 6 provides a numerical example of the welfare changes under short-term and long-term debt. Concluding remarks are outlined in Section 7. Proofs are relegated to the Appendix.

2 Baseline Model

2.1 Setup

The following model setup and notation follows Cole and Kehoe (2000) very closely. Consider a government in a small open economy inhabited by a continuum of identical households. Households live infinite periods and are uniformly distributed over the $[0,1]$ interval. The government maximizes households' utility

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(g_t\right),$$

where g_t is government spending, $\beta \in (0,1)$ the discount factor, and $u(\cdot)$ satisfies $u'(\cdot) > 0$, $u''(\cdot) < 0$ and $u(0) \to -\infty$.

The economy receives a deterministic sequence of endowments (see Conesa and Kehoe (2017)),

$$Z^{1-z_t} y = \begin{cases} y, & z_t = 1 \\ Zy, & z_t = 0 \end{cases},$$

where y is a positive constant, $Z \in (0,1)$ is the share of output that is left after a default episode, and z_t reflects the default decision of the government. When the government defaults on its obligations, $z_t = 0$; otherwise, z_t stays at 1. In the state space, z_{-1} reflects whether the government defaulted on its debt before; if that happened, z_{-1} remains at 0 forever. The government has access to the international credit market only if it did not default in the past. The international credit market is competitive, and has risk-neutral lenders who buy bonds and have finite wealth. Lenders and the government take the price of a bond q_t as given. If the government never defaulted before, it starts each period with a positive amount of maturing debt B_t. Thereafter, the government chooses g_t, sells B_{t+1} one-period bonds at price q_t to finance its expenditures, and decides whether to repay or not its maturing debt B_t.

The budget constraint of the government is

$$g_t + z_t B_t = Z^{1-z_t} y + q_t B_{t+1}.$$

Agents in the model observe the realization of a sunspot ζ_t in every period. This random variable ζ_t is independent and identically distributed with uniform CDF on the unit interval $[0, 1]$. The sunspot, together with the current maturing debt and the government past default decisions, define the aggregate state s_t in this economy, $s_t = (B_t, z_{-1}, \zeta_t)$.

The timing within a period is as follows. The government and the creditors observe the aggregate state of the economy s_t. Taking the price q_t as given and if no default occurred before, the government issues B_{t+1} bonds and the creditors purchase that quantity. At the end, the government makes its default and spending decisions, z_t and g_t. It is worth noting that the default decision occurs after borrowing in this timing, in contrast to models of sovereign default where the borrowing decision occurs after default and impede belief-driven crises.(see Arellano, 2008; Arellano and Ramanarayanan, 2012; Hatchondo and Martinez, 2009)

2.2 An equilibrium with short-term debt

This section describes the government value function and the price of debt that define an equilibrium in the model.

Government value function. Given the aggregate state s_t, the government chooses g_t, B_{t+1}, and z_t in two stages within each period. Conditional on no previous default, the value when the government first decides how much to borrow from the international credit market is given by

$$V(s) = \max_{B'} u(g) + \beta E V(s') \tag{1}$$

$$\begin{aligned} s.t. \quad g + zB &= Z^{1-z} y + q(s, B') B' \\ z &= z(s, B', q(s, B')) \\ g &= g(s, B', q(s, B')) \\ s' &= (B', z, \zeta'), \end{aligned}$$

with optimal policy function $B'(s)$. When $B'(s)$ increases, default becomes more likely and the equilibrium price decreases. This drop in q will act as a no-Ponzi-game condition in problem (1), preventing large borrowings from the government.

In the second stage, the government makes its default and spending decisions,

$$\max_{g,z} u(g) + \beta EV(s') \tag{2}$$

$$\begin{aligned} s.t. \quad g + zB &= Z^{1-z}y + q(s,B')B' \\ z &= 1 \text{ or } z = 0 \\ s' &= (B', z, \zeta'). \end{aligned}$$

It is immediate to observe that policy functions $z(s,B',q(\cdot))$ and $g(s,B',q(\cdot))$ from problem (2) depend on the choice of B'. In the next sections, we will use this feature to solve the government problem using a backward induction logic.

Price of debt. The government borrows money from a competitive international credit market. Creditors in this market discount at the same rate as the government, and $\beta = 1/(1+r)$, with r being a fixed interest rate. Hence, the equilibrium price satisfies

$$q(s,B') = \beta E[z'((s', B'(s'), q(s', B'(s'))))]. \tag{3}$$

The price of a one-period bond equals the inverse of the gross interest rate, adjusted by tomorrow's expected probability of default, $E[z'(\cdot)]$.

Using eqs. (1)-(3), an equilibrium under short-term debt is defined as follows:

Definition 1 (Short-term debt) *Given an initial stock of debt B_0, and a given repayment decision from the past z_{-1}, a recursive equilibrium with short-term debt is a government value function $V(s)$; government policy functions $B'(s)$, $g(s,B',q(\cdot))$, and $z(s,B',q(\cdot))$; and a price function $q(s,B')$ such that:*

1. *Given $g(s,B',q(\cdot))$, $z(s,B',q(\cdot))$ and $q(s,B')$, the government policy function $B'(s)$ maximizes the value function $V(s)$ represented by the government problem (1).*

2. *Given the value function $V(s)$ and the price $q(s,B')$, policy functions $g(s,B',q(\cdot))$ and $z(s,B',q(\cdot))$ solve the second government's problem represented by equation (2).*

3. *The price $q(s,B')$ at which the government borrows from the international credit market satisfies (expected) zero profits of competitive risk-neutral creditors (equation (3)).*

3 The Crisis region under short-term debt

We now define different regions of debt related to the previous definition of a sunspot equilibrium. Then, we present two conditions defining bounds of these regions. Finally, we provide a numerical illustration of these regions and describe optimal debt policies for the government. Our exposition (including propositions, proofs and figures) follow Cole and Kehoe (2000) very closely.

3.1 Debt regions

Two thresholds $\underline{B}$ and $\bar{B}$ characterize debt regions with different equilibrium outcomes for the model economy.[1] For debt levels B below $\underline{B}$, the No-Default region defines an interval of debt where the government always chooses to repay its obligations. In contrast, debt levels B above $\bar{B}$ define a Default region where the government optimally chooses to default. Finally, starting from a debt level $B \leq \underline{B}$ and as B increases, debt levels pass through an interval $\left[\underline{B}, \bar{B}\right]$ called the Crisis region where both equilibrium outcomes can occur with positive probability. When debt lies in this region, creditors expect default to occur with a fixed probability π, and the sunspot ζ becomes a coordination device among them. Specifically, if $\zeta \leq \pi$, the sunspot signals that the government will default, and therefore creditors stop lending. As a result, q drops to 0, and the government defaults because it cannot roll over its debt. Conversely, if $\zeta > \pi$, creditors expect the government to repay, and therefore q remains positive and provides liquidity to the government.

3.2 The optimal price q under short-term debt

If the government always repaid (i.e., $z_{-1} = 1$), then the government and creditors trade bonds at the equilibrium price

$$q\left(\left(B, 1, \zeta\right), B'\right) = \begin{cases} \beta, & \text{if } B' \leq \underline{B} \\ \hat{\beta}, & \text{if } \underline{B} < B' \leq \bar{B} \\ 0, & \text{if } \bar{B} < B' \end{cases}, \tag{4}$$

with $\hat{\beta} = \beta\left(1 - \pi\right)$. The equilibrium price (4) reflects the government choices of B' (the first decision of the government). When B' is above $\bar{B}$, creditors expect default to occur with probability 1, and therefore $q = 0$. When B' is below $\underline{B}$, creditors expect the government to

[1] Bound $\bar{B}$ will depend on π, i.e. $\bar{B} = \bar{B}\left(\pi\right)$. For notational convenience, we omit this dependence whenever possible.

default with probability 0, and $q = \beta$. Otherwise, when tomorrow's debt lies within $\left[\underline{B}, \bar{B}\right]$, creditors expect default to occur with a fixed probability π, implying that $q = \hat{\beta}$.[2]

3.3 Determining the region boundaries

We now present the conditions that characterize both thresholds $\underline{B}$ and $\bar{B}$. Following the literature, we take $t = 0$ to be the first period.

The No-Lending Condition (NLC) states that the government strictly prefers to default instead of repaying maturing debt when the price of debt is 0. Analytically, this implies

$$V^D(s, 0, 0) > V^R(s, 0, 0), \tag{5}$$

where $V^R(\cdot)$ is the value of repayment and $V^D(\cdot)$ is the value function under default. Intuitively, the NLC suggests that the government defaults instead of repaying debt when it does not obtain any liquidity from the credit market. Therefore, when $q = 0$, the government choice of B' becomes trivial and assumed to be 0. Finally, the equality of equation (5) characterizes the lower bound $\underline{B}$ of the Crisis region.

The Participation Constraint (PC) states the condition for the government to always prefer repayment versus default,

$$V^R(s, B', q) \geq V^D(s, B', q). \tag{6}$$

This expression states that the government prefers to honour its debts when the value of repayment $V^R(\cdot)$, is greater than defaulting and becoming excluded from the credit market, $V^D(\cdot)$. The equality of equation (6) characterizes the upper bound $\bar{B}$ of the Crisis region, and the government prefers default at any debt level above $\bar{B}$.

3.4 Characterization of the lower bound of the Crisis region

If the government chooses to default at t, the government budget constraint in that period is

$$g = Zy + q(s, B')B'.$$

After default, the government only finances spending with a share Z of total output y, and loses access to the international credit market. Therefore, the budget constraint from $t + 1$

[2] See Section A.1 of the Appendix for the calculation of equation (4).

onward is simply $g = Zy$. The value of default for any price q is therefore given by

$$V^D(s, B', q) = u(Zy + q(s, B')B') + \beta\frac{u(Zy)}{1-\beta}. \tag{7}$$

Setting $q = 0$ in the last expression leads to the left-hand side of the No-Lending Condition (5),

$$V^D(s, 0, 0) = \frac{u(Zy)}{1-\beta}. \tag{8}$$

To compute the right-hand side of the NLC, the budget constraint at t when the government still repays but $q = 0$ is $g = y - B$. Thereafter, the government finances spending with total output (i.e., $g = y$) and $V^R(\cdot)$ in the No-Lending Condition (5) renders

$$V^R(s, 0, 0) = u(y - B) + \beta\frac{u(y)}{1-\beta}. \tag{9}$$

After replacing eqs. (8) and (9) in the NLC (5), the lower bound $\underline{B}$ of the Crisis region is the debt level such that expression

$$\frac{u(Zy)}{1-\beta} > u(y - B) + \beta\frac{u(y)}{1-\beta} \tag{10}$$

holds with equality. In Section $A.2$ of the Appendix, Lemma 6 characterizes the existence and uniqueness of $\underline{B}$.

3.5 Characterization of the upper bound $\bar{B}$ of the Crisis region

The right-hand side of the Participation Constraint (6) follows immediately from equation (7). When government debt B' lies in the Crisis region, the value of defaulting today is

$$V^D(s, B', q) = u\left(Zy + \hat{\beta}B'\right) + \beta\frac{u(Zy)}{1-\beta},$$

where the equilibrium price of debt is now $q = \hat{\beta}$.

To obtain the left-hand side of the Participation Constraint (6), the value of repayment depends on different paths of borrowing that the government can choose in the Crisis region. In particular, the government can decide to lower debt in T arbitrary periods, or also choose to never run down debt, i.e. $T \to \infty$. We will therefore compute $V^R(s, B', q)$ describing the optimal choice of $\{B_{t+1}\}$ by dividing the government problem in two parts, first when government debt is in the Crisis region, and second when debt is in the No-Default region. Specifically, in the first part, initial debt B_0 starts in the Crisis region at $t = 0$, and the government exits the Crisis region in T periods by choosing an optimal path $\{B_{t+1}\}_{t=0}^{T-1}$. In the second part, the government begins already at the No-Default region, and starting from $t = T$ it will choose the optimal path of debt $\{B_{t+1}\}_{t=T}^{\infty}$. In what follows, we use a backward induction logic and start from the second problem.

Assume that the government never defaulted in the past ($z_{-1} = 1$). The government problem after reaching the No-Default region in T periods and starting with an initial debt level $B_T = \underline{B}$ is given by

$$\max_{\substack{\{B_{s+1}\}_{s=T}^{\infty}, \\ \{g_s\}_{s=T}^{\infty}}} \sum_{s=T}^{\infty} \beta^s u(g_s) \tag{11}$$

$$\begin{aligned} s.t. \quad & g_s + B_s = y + q_s B_{s+1} \quad \forall s \geq T \\ & B_T = \underline{B} \quad given \\ & B_s \leq \underline{B} \quad \forall s \geq T. \end{aligned}$$

(Refer to Section $A.3$ of the Appendix for the full solution.)

The Euler Equation

$$u'(g_t) = u'(g_{t+1})$$

states that the government smooths g_t across periods and attains a constant level of spending

$$\bar{g} = y - (1-\beta)\underline{B}.$$

Since the government's optimal choice of debt remains at $\underline{B}$ for every period $t = T, T+1, T+2, \ldots$, the present value from smooth spending renders

$$\frac{u(y - (1-\beta)\underline{B})}{1-\beta},$$

and represents the government's continuation payoff after reaching the No-Default region.

Equipped with the previous result, we now turn to the first part of the government problem. Starting at $t = 0$ and with a given debt level B_0 in the Crisis region, the government lowers debt in T periods to reach the No-Default region. Hence, the total payoff $V^T(B_0)$ from lowering debt in $T \geq 2$ periods is[3]

$$V^T(B_0) = \max_{\substack{\{B_{t+1}\}_{t=0}^{T-1}, \\ \{g_t\}_{t=0}^{T-1}}} \sum_{t=0}^{T-1} \hat{\beta}^t u(g_t) + \hat{\beta}^{T-1} \beta \frac{u(y - (1-\beta)\underline{B})}{1-\beta} + \left[\frac{1-\hat{\beta}^{T-1}}{1-\hat{\beta}}\right] \beta\pi \frac{u(Zy)}{1-\beta} \tag{12}$$

[3] Refer to the Appendix for the full calculation of $V^T(B_0)$ (Section $A.3.2$), the Euler Equation (Section $A.3.3$), and the calculation of $g^T(B_0)$ (Section $A.3.4$). Note that when $T = 1$, the government jumps from B_0 to $\underline{B}$ and exits the Crisis region. Since no intermediate choice of $\{B_{t+1}\}$ takes place during the transition, we omit writing this case here, and relegate its details to the Appendix (Section $A.3.2$).

$$
\begin{aligned}
s.t. \quad & g_t = y - B_t + q_t B_{t+1} \quad \forall t = 0, 1, ..., T-2 \\
& g_t = y - B_t + q_t \underline{B} \quad t = T-1 \\
& q_t = \beta E_t \left[z_{t+1} \right] \\
& B_0, \quad \underline{B} \; given, \qquad z_{-1} = 1.
\end{aligned}
$$

The first two terms in equation (12) show the government's payoff when the sunspot does not trigger government default. With probability $1-\pi$, the (discounted) value of lowering B_{t+1} yields a utility of $u(g_t)$ in each period $t \leq T-1$, and the government achieves the continuation payoff $u(y-(1-\beta)\underline{B})$ from $t \geq T$ onward. Conversely, the last term in equation (12) captures the payoff from following a policy T when the sunspot triggers government default. Specifically, the government is subject to the realization of a self-fulfilling debt crisis with probability π in every period it remains in the Crisis region. When that is the case, it can only spend Zy.

Maximization of expression (12) renders the same Euler Equation as before, $u'(g_t) = u'(g_{t+1})$, implying that government spending is constant during the transition. Following the literature, $g_t = g^T(B_0)$ denotes the optimal level of government spending given B_0 and the chosen policy T. Replacing each $g^T(B_0)$ in the T budget constraints, the optimal level $g^T(B_0)$ is given by

$$
g^T(B_0) = y - \left[\frac{1-\hat{\beta}}{1-\hat{\beta}^T} \right] B_0 + \left[\frac{1-\hat{\beta}}{1-\hat{\beta}^T} \right] \hat{\beta}^{T-1} \beta \underline{B}, \tag{13}
$$

and the value function associated to problem (12) when the government runs down debt in T periods is

$$
V^T(B_0) = \left[\frac{1-\hat{\beta}^T}{1-\hat{\beta}} \right] u\left(g^T(B_0)\right) + \hat{\beta}^{T-1} \beta \frac{u(\bar{g})}{1-\beta} + \left[\frac{1-\hat{\beta}^{T-1}}{1-\hat{\beta}} \right] \beta\pi \frac{u(Zy)}{1-\beta}. \tag{14}
$$

This last expression (14) captures the value function for any policy T. Additionally, taking the limit as $T \to \infty$ in expressions (13) and (14) yields the value when the government never lowers initial debt B_0,

$$
V^\infty(B_0) = \frac{u\left(y - \left(1-\hat{\beta}\right) B_0\right)}{1-\hat{\beta}} + \frac{\beta\pi u(Zy)}{(1-\beta)\left(1-\hat{\beta}\right)}.
$$

Finally, we define the maximum across every payoff $V^T(B_0)$ when $T = 1, 2, ..., \infty$ as the left-hand side of the Participation Constraint (6), and obtain the final expression of the PC as

$$\max\left\{V^1(B_0), V^2(B_0), ..., V^\infty(B_0)\right\} \geq u\left(Zy + \hat{\beta}B_0\right) + \beta\frac{u(Zy)}{1-\beta}. \tag{15}$$

As before, the upper bound of the Crisis region is the debt level $\bar{B}$ such that expression (15) holds with equality.

3.6 Crisis equilibria under short-term debt

From the last section, equations (10) and (15) characterized the lower and the upper bounds of the Crisis region respectively. In what follows, we numerically show Crisis regions for different default probabilities. With a similar approach to Cole and Kehoe (2000), we first compute the Crisis region in an equilibrium where the sunspot channel is switched off, i.e., $\pi = 0$. This equilibrium is called a zero-probability crisis equilibrium. Then, we compute a Crisis region under strictly positive creditors' beliefs π. Finally, we present analytical and numerical results on the government optimal debt policies.[4]

A zero-probability crisis equilibrium. When creditors' belief of default π becomes smaller, the government's incentives to run down debt decrease. Intuitively, the government must prefer to roll over its total initial debt when lenders never expect default, i.e. $\pi \to 0$. Analytically, this implies that V^∞ should strictly dominate any other payoff V^T (with $T < \infty$) in the LHS of the Participation Constraint (15). In Section $A.3.5$ of the Appendix, Lemma 8 proves this argument, and therefore the Participation Constraint (15) under a zero-probability crisis equilibrium becomes

$$\frac{u(y-(1-\beta)B_0)}{1-\beta} \geq u(Zy + \beta B_0) + \beta\frac{u(Zy)}{1-\beta}. \tag{16}$$

Finally, Lemmas 6 and 9 establish existence and uniqueness of debt levels $\underline{B}$ and $\bar{B}$ satisfying eqs. (10) and (16) at equality.

Rewriting the No-Lending Condition (10) and the Participation Constraint (15) as

$$U_{NLC}(B) \equiv u(y-B) + \beta\left[\frac{u(y)}{1-\beta}\right] - \frac{u(Zy)}{1-\beta}$$

$$U_{PC}^{\pi=0}(B) \equiv \frac{u(y-(1-\beta)B)}{1-\beta} - u(Zy+\beta B) - \beta\frac{u(Zy)}{1-\beta},$$

we observe that as debt levels increase, both curves monotonically decrease and cross the

[4] For ease of exposition, we will focus on stationary debt policies when characterizing the Crisis region. For further details on the stationarity of debt policies, refer to the discussion in Cole and Kehoe (2000).

0-line once. Section A.3.6 of the Appendix analyzes functions $U_{NLC}(B)$ and $U_{PC}^{\pi=0}(B)$ on the domain of B.

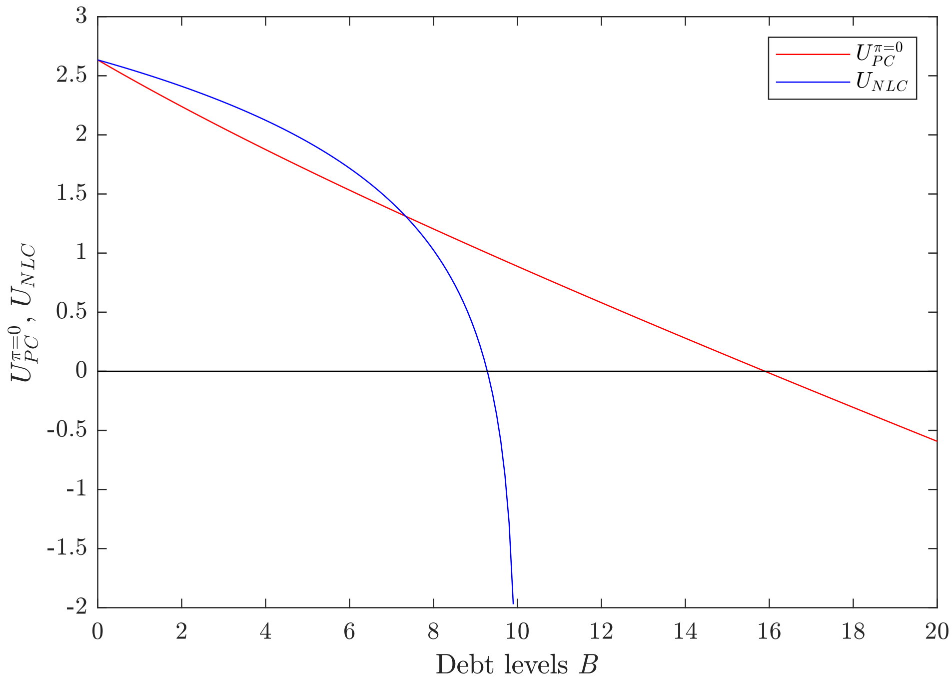


Fig. 2. Crisis region when $\pi = 0$, short-term debt. ($\pi = 0$, $\beta = 0.96$, $Z = 0.9$, $y = 10$ and $u(\cdot) = \ln(\cdot)$.)

Figure 2 plots the $U_{PC}^{\pi=0}(B)$ and the $U_{NLC}(B))$ as B increases. The government utility is $u(g_t) = \log(g_t)$. In addition, constant output y is set at 10, while the share of remaining output after default is $Z = 0.9$. The discount parameter is $\beta = 0.96$, and the probability of default π is set to 0. The Crisis region is characterized by the curves $U_{PC}^{\pi=0}(B)$ and $U_{NLC}(B)$ where debt levels B satisfy $U_{PC}^{\pi=0}(B) \geq 0 \geq U_{NLC}(B)$. In the particular example of Figure 2, the interval of debt levels where the sunspot ζ_t triggers self-fulfilling debt crises (with probability $\pi = 0$) is $B \in [9.28, 15.9]$.

A positive-probability crisis equilibrium. We now compute a nonempty Crisis region where self-fulfilling debt crises occur with positive probability, i.e. $\pi > 0$. Using $V^{\infty}(B_0)$, we rewrite the Participation Constraint (15) as

$$U_{PC}^{\pi>0}(B) \equiv \frac{u\left(y - \left(1 - \hat{\beta}\right)B\right)}{1 - \hat{\beta}} + \frac{\beta\pi}{1 - \hat{\beta}}\frac{u(Zy)}{1 - \beta} - u\left(Zy + \hat{\beta}B\right) - \beta\frac{u(Zy)}{1 - \beta}.$$

In Section $A.3.7$ of the Appendix, we show that $U_{PC}^{\pi>0}(B)$ starts positive at $B = 0$ and monotonically decreases crossing the 0 line as B rises. Since the No-Lending Condition

(10) is independent from π, $U_{NLC}^{\pi>0}(B) = U_{NLC}(B)$ $\forall \pi \in [0,1]$, and therefore has the same behavior as before.

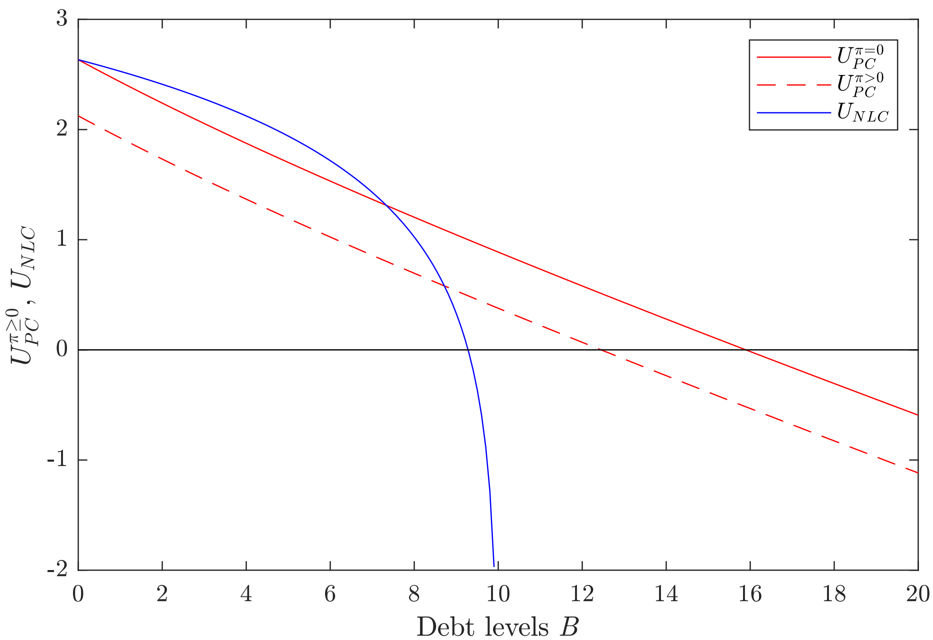


Fig. 3. Crisis region when $\pi = 0$ and $\pi > 0$, short-term debt. ($\pi = \{0, 0.01\}$, $\beta = 0.96$, $Z = 0.9$, $y = 10$ and $u(\cdot) = \ln(\cdot)$.)

As creditor's default belief π increases, Figure 3 illustrates how the upper bound $\bar{B}(\pi)$ decreases relative to $\bar{B}(0)$. In Section A.3.8 of the Appendix, Lemma 11 analytically proves that $d\bar{B}(\pi)/d\pi < 0$. This result implies that some debt levels below the upper bound $\bar{B}(0)$ strictly violate the Participation Constraint (15) when $\pi > 0$, thus making $\bar{B}(\pi) < \bar{B}(0)$. Intuitively, when π rises, creditors believe that some debt levels are now too large to be honoured, thus expecting the government to default for sure at these debt levels. Also, the government now prefers to default at large debt levels B and collect $\hat{\beta}B$, instead of facing creditors' panic with probability π and collect 0. These interaction ultimately increases the interval where the government defaults with certainty, lowering the upper bound $\bar{B}$ of the Crisis region.

Moreover, when $\pi > 0$ and initial debt is not very large, it is intuitive that the government optimally evaluates policies to escape the Crisis region. As the next Proposition shows, the government will choose different optimal policies for $\{B_{t+1}\}$, depending on the initial debt when $\pi > 0$. This result has been established by Cole and Kehoe (2000) and is one of the most relevant findings in their paper. In Section A.3.9 in the Appendix, we analytically prove the most important parts of their Proposition applied to our version of the model.

Proposition 2 *Let $V^T(B_0)$ be the government's payoff when its policy is to lower its debt to $\underline{B}$ in T periods. Then for any $B_0 < B(\pi)$ there exists a $T^*(B_0) \in \{1, 2, ..., \infty\}$ that maximizes $\{V^1(B_0), V^2(B_0), ..., V^\infty(B_0)\}$. Moreover, for π close to 0, there are necessary regions of B_0 with the full range of possibilities $T^*(B_0) = 1, 2, ..., \infty$.*

Proof. See Appendix. ■

When $\pi > 0$ and B_0 lies in the Crisis region, government's default and repayment decisions ultimately depend on the realization of the random variable ζ_t. When this is the case, the result in Proposition 2 states that the government optimally chooses to exit the Crisis region in $T^*(B_0)$ periods. Furthermore, this result also states that for a small positive π, the Crisis region can be divided into different intervals of debt where the government optimally sets a different $T^*(B_0)$, with $T^*(B_0) = 1, 2, ..., \infty$. To understand the intuition behind, recall that the government knows that the sunspot triggers default when $B_0 \in [\underline{B}, \bar{B}(\pi)]$. Proposition 2, therefore, states that the government can optimally choose to exit the Crisis region to avoid being subject to creditors' beliefs.

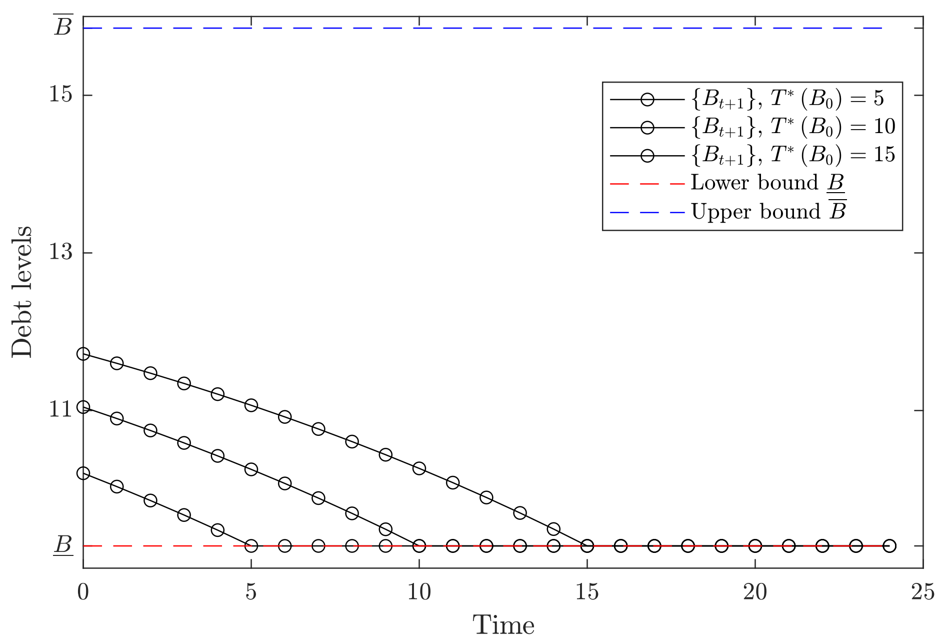


Fig. 4. Optimal $\{B_{t+1}\}$ in the Crisis region $[\underline{B}, \overline{B}]$, at different B_0 and associated $T^*(B_0$ short-term debt. ($\pi = 0.0001$, $\beta = 0.96$, $Z = 0.9$, $y = 10$ and $u(\cdot) = \ln(\cdot)$.)

At a different initial B_0 in the Crisis region, Figure 4 depicts three optimal debt policies when self-fulfilling debt crises occur with positive probability. When B_0 is close to $\underline{B}$, the government lowers debt faster than when B_0 approaches $\bar{B}$ – see, for example, the lower optimal path for $\{B_{t+1}\}$ when $T^*(B_0) = 5$. As B_0 increases, government debt is further from the No-Default region, and Proposition 2 establishes that the optimal $T^*(B_0)$ does not decrease. The government then faces the following trade-off: on the one hand, the fastest it reaches the No-Default region, the lower the probability that the sunspot triggers default. On the other hand, a sharp decline in debt B_{t+1} leads to lower government spending. Furthermore, initial per-period utility levels $u(\cdot)$ are more valuable because of discounting. To balance these gains and losses when B_0 rises, Proposition 2 states that the government can only delay its exit from the Crisis region, i.e. increase $T^*(B_0)$. Examples of these optimal choices are depicted by the middle and uppermost paths of $\{B_{t+1}\}$, with $T^*(B_0) = 10$ and $T^*(B_0) = 15$ respectively.

To better illustrate these effects, we take the difference between two value functions at T and $T+1$ for a given $B_0 \in \left(\underline{B}, \bar{B}(\pi)\right)$,

$$
\begin{aligned}
V^T(B_0) - V^{T+1}(B_0) &= \frac{1-\hat{\beta}^T}{1-\hat{\beta}}\left[u\left(g^T(B_0)\right) - u\left(g^{T+1}(B_0)\right)\right] + \hat{\beta}^T\left[u(\bar{g}) - u\left(g^{T+1}\right)\right] \\
&\quad + \frac{\hat{\beta}^{T-1}\beta\pi}{1-\beta}\left[u(\bar{g}) - u(Zy)\right].
\end{aligned}
$$

The first term in brackets represents the loss associated to plan T versus $T+1$. In Section $A.3.10$, Claim 13 shows that $g^T(B_0) < g^{T+1}(B_0)$. This means that government spending is higher when the government smooths g across more periods. Intuitively, when the government delays exiting the Crisis region from T to $T+1$, it allocates the decrease of initial B_0 into more periods. This implies that government spending is relatively higher at $T+1$ than at T. In contrast, the second and third terms represent the government gains from choosing T. Lemma 12 in Section A.3.9 in the Appendix shows that spending $\bar{g}$ is always bigger than g^T for any finite T. In particular, $u(\bar{g}) - u(g^{T+1})$ is the benefit from attaining the No-Default region earlier given policy T (which is faster than $T+1$). Finally, the difference $u(\bar{g}) - u(Zy)$ captures the relative gain once government spending reaches the constant level $\bar{g}$. The higher is $u(\bar{g})$ relative to $u(Zy)$, the higher are the government's incentives to exit the Crisis region soon. Not surprisingly, this term increases as the transition T becomes shorter: the earlier the government arrives at $\underline{B}$, the lower will be the impact of discounting on this term.

Some comparative statics. We analyze how the amount of output left after a default, Z, affects the bound of the Crisis region and the government value function.

Proposition 3 *An increase in Z (i) strictly decreases both bounds $\underline{B}$ and $\bar{B}(\pi)$ of the Crisis region, (ii) strictly decreases the value of running down debt in one period $V^1(B_0)$, and (iii)* *strictly increases the value of debt roll over $V^{\infty}(B_0)$.*

Proof. See Appendix. ■

Result (i) states that the lower and upper bounds of the Crisis region, $\underline{B}$ and $\bar{B}$, are decreasing in Z,

$$\frac{\partial \underline{B}}{\partial Z} = -\frac{\frac{u'(Zy)}{1-\beta} y}{u'\left(y - \underline{B}\right)} < 0 \quad , \quad \frac{\partial \bar{B}(\pi)}{\partial Z} = -\frac{\beta \frac{u'(Zy)y}{1-\beta}\left[1 - \frac{\pi}{1-\hat{\beta}}\right] + u'\left(Zy + \hat{\beta}\bar{B}\right) y}{u'\left(y - \left(1-\hat{\beta}\right)\bar{B}\right) + u'\left(Zy + \hat{\beta}\bar{B}\right)\hat{\beta}} < 0.$$

To understand the intuition behind the first expression, assume that $Z_0 < Z_1$. When Z rises from Z_0 to Z_1, a default event reduces the economy's output by less. This, in turn, improves the default outcome for the government for every debt level. In response to that, rational creditors have to attribute a default probability π to more debt levels below the initial $\underline{B}(Z_0)$. Intuitively, creditors must expect that the government has less incentives to repay after the government's default scenario becomes better. Therefore, the lower bound of the Crisis region needs to decrease to define a new threshold $\underline{B}(Z_1)$, such that $\underline{B}(Z_1) < \underline{B}(Z_0)$ – below which creditors again expect repayment with probability 1 again.

A similar logic applies to the upper bound of the Crisis region. A rise in Z would only improve the default scenario for the government. Since this increases the value V^D, the government has now greater incentives to default for any debt level. But if $\bar{B}(\pi)$ remained unchanged, creditors would still expect repayment of some levels of debt with probability $1-\pi$, where the government now strictly prefers default. Therefore, rational creditors must adjust their beliefs and assign probability $1-\pi$ of repayment to lower debt levels, thus decreasing $\bar{B}(\pi)$ after Z rises.

For part (ii), observe that

$$\frac{dV^1}{dZ} = \beta \frac{\partial \underline{B}}{\partial Z}\left[u'(g_0) - u'(\bar{g})\right] < 0.$$

By (i), recall that $\partial \underline{B}/\partial Z < 0$. Hence, an increase in the output left post-default also increases the distance between the initial stock of debt B_0 and the lower bound of the No-Default region, $\underline{B}(Z_1)$. Because of this, debt levels B_{t+1} have to be reduced by a larger amount in order to attain the (now further) bound $\underline{B}(Z_1)$. As a result, this effect lowers total payoff $V^1(B_0)$.

Part (iii) of the Proposition argues that when the optimal plan entails never running down debt, the government's payoff responds positively to an increase in the productivity parameter Z.

$$\frac{dV^{\infty}(B_0)}{dZ} = \frac{\beta\pi u(Zy)}{\left(1-\hat{\beta}\right)(1-\beta)}y > 0$$

The last term of expression $V^{\infty}(B)$ indicates that if the government is entitled to permanently keep its initial debt level, then it will always be vulnerable to self-fulfilling debt crises. If the government is likely to default in every period, then the government's payoff can only rise if the bad outcome Zy increases.

4 A model with long-term debt

In this Section, we present the law of motion of long-term debt issuances. We then show the government problem that enters into the new definition of a sunspot equilibrium under long-term debt. In Section 6, we show the associated Crisis region and government optimal choices of debt when bonds have long maturity.

4.1 Law of motion of long-term debt and new budget constraint

In the previous sections, we presented Crisis regions for $\pi = 0$ and $\pi > 0$, and described optimal government policies under short-term debt. Here, we study how the above results change after switching to long-term debt defined as in Hatchondo and Martinez (2009).

In this new set-up, i_t represents the issuance of long-term bonds at any period t. At period 0, the government now starts with a given amount of outstanding bonds issued from the past, i_{-1}, and in every period t the government collects $q_t i_t$ after issuing i_t bonds. Also, recall that under short-term debt, total outstanding debt was repaid in $t+1$. In contrast to this case, under long-term debt the government now repays coupons worth δ^{n-1} in every period $t+n$ and $n \geq 1$, where $\delta \in (0,1)$ denotes the rate of geometric decrease of coupon payments across time.

Conditional on no default, the government budget constraints for periods $0, 1, ..., t$ are

$$\begin{aligned}
&g_0 + i_{-1} = y + q_0 i_0 \\
&g_1 + \delta i_{-1} + i_0 = y + q_1 i_1 \\
&\quad\vdots \\
&g_t + \delta^t i_{-1} + \delta^{t-1} i_0 + ... + \delta i_{t-2} + i_{t-1} = y + q_t i_t.
\end{aligned}$$

Calling $B_t = \sum_{j=0}^{t-1} \delta^{t-j-1} i_j + \delta^t i_{-1}$ the stock of maturing debt (i.e., the total coupons due at

t from past issuances), B_0 equals i_{-1} and the budget constraint becomes

$$g_t + z_t B_t = Z^{1-z_t} y + q_t (s_{t,} B_{t+1})\, i_t \quad \forall t. \tag{17}$$

In addition, the law of motion describing i_t is

$$i_t = B_{t+1} - z_t B_t \delta. \tag{18}$$

What equation (18) states is that (i) debt issuances depend on the default choice z_t, and (ii) i_t units will become part of the stock of coupon payments B_{t+1} that are due tomorrow. Finally, note that when δ equals 0, government issuance is $i_t = B_{t+1}$ – and we return to the short-term debt set-up.

4.2 Government value function and new price of debt

Under long-term debt, the choice of i_t occurs in stage 2 of the model, and equation (18) becomes an additional constraint to the government problem (1). Given state s, the first government decision involves choosing B' and i according to

$$V(s) = \max_{i,B'} u(g) + \beta E V(s') \tag{19}$$

$$\begin{aligned} s.t. \quad g + zB &= Z^{1-z} y + q(s, B')\, i \\ i &= B' - zB\delta \\ z &= z(s, B', q(s, B')) \\ g &= g(s, B', q(s, B')) \\ s' &= (B', z, \zeta'). \end{aligned}$$

Since the second government decision remains unchanged, problem (2) depicts the choice of z and g.

Government and creditors trade long-term debt in a competitive international credit market at the equilibrium price

$$q(s, B') = \beta E\left[\left[1 + q(s', B'(s'))\,\delta\right] z'(s', B'(s'), q(s', B'(s')))\right]. \tag{20}$$

Similar to the one-period bond case, the first term of the sum represents the value of a coupon received tomorrow. The difference is now that a bondholder receives a stream of coupons in every period t. Therefore, the second term can be interpreted as the value of reselling the long-term bond after it paid its first coupon.

We now define the equilibrium under long-term debt.

Definition 4 (Long-term debt) *Given an initial stock of debt B_0, and a given repayment decision from the past z_{-1}, a recursive equilibrium with long-term debt is a government value function $V(s)$; policy functions $B'(s)$, $i(s)$, $g(s, B', q(\cdot))$, and $z(s, B', q(\cdot))$; and a price function $q(s, B')$ such that:*

1. *Given $g(s, B', q(\cdot))$, $z(s, B', q(\cdot))$ and the price $q(s, B')$, the government policy functions $B'(s)$ and $i(s)$ maximize the value function $V(s)$ represented by the government problem (19).*

2. *Given the value function $V(s)$ and the price $q(s, B')$, policy functions $g(s, B', q(\cdot))$ and $z(s, B', q(\cdot))$ solve the second government's problem represented by eq. (2).*

3. *The price at which the government borrows from the international credit market (20) satisfies (expected) zero profits of competitive risk-neutral creditors.*

5 The Crisis region under long-term debt

The definition of debt regions outlined in Section 3 holds for long-term debt. In the next parts, we present new features of the price of debt, as well as the value functions that define the bounds of the Crisis region.

5.1 Optimal price under long-term debt and determination of the region bounds

In what follows, we particularize the equilibrium price of debt (20) when debt B' lies in different debt regions. When $B' \leq \underline{B}$, the international credit market expects no default and, hence, the government always repays. The equilibrium price is then stationary and equal to

$$q(s, B') = \frac{\beta}{1 - \beta\delta}.$$

When debt lies in the Crisis region, the international credit market expects the government to default with probability π. Under debt roll over, the equilibrium price is again stationary but now yields

$$q(s, B') = \frac{\hat{\beta}}{1 - \hat{\beta}\delta}.$$

Using a backward induction argument, we now illustrate the logic when the government runs down debt in $T = 2$ periods – we relegate to the Appendix the general case when T is any arbitrary number. Be $q_t^T(s, B') \equiv q_t^T$ the current price at t when exiting the Crisis region takes T periods. Starting at $t = 0$, in $t = 1, 2$ the optimal price does not depend on the realization of the sunspot ζ_t since creditors know that tomorrow's choice of debt lies in the No-Default region. Hence, $q_1^2 = q_2^2 = \beta/(1 - \beta\delta)$. However, given that the government planned to exit in $T = 2$ periods, at $t = 0$ creditors know that the government debt will lie in the Crisis region in $t = 1$. Thus, at $t = 0$ the optimal price of debt is

$$q_0^2 = \beta E\left[\left(1 + q_1^2\delta\right) z'\right] = \hat{\beta} + \hat{\beta}\delta\frac{\beta}{1 - \beta\delta}.$$

Following this logic, the equilibrium price when the government exits the Crisis region in T arbitrary periods is[5]

$$q(\cdot) = \begin{cases} \frac{\beta}{1-\beta\delta} & \text{, if } B' \leq \underline{B} \\ (1 - (\hat{\beta}\delta)^{T-1-t})\frac{\hat{\beta}}{1-\hat{\beta}\delta} + (\hat{\beta}\delta)^{T-1-t}\frac{\beta}{1-\beta\delta} & \text{, if } \underline{B} < B' \leq \bar{B} \text{ and } t < T-1 \\ 0 & \text{, if } \bar{B} < B'. \end{cases} \tag{21}$$

The expression shows that price q_t^T is now the convex combination of the optimal prices $\beta/(1 - \beta\delta)$ and $\hat{\beta}/(1 - \hat{\beta}\delta)$. In particular, the weights depend on both the government choice of T, and the time left to exit the Crisis region. Finally, expression (21) is consistent with the price being stationary when the government always rolls over debt that lies in the Crisis region. When $T \to \infty$, the second term vanishes since the international credit market expects the government to never lower debt, and hence $q_t^\infty = \hat{\beta}/(1 - \hat{\beta}\delta)$ for any t.

In what follows, we denote $\underline{B}$ and $\bar{B}$ as $\underline{B}(\delta)$ and $\bar{B}(\delta)$ in order to highlight the role of debt maturity in the bounds of the Crisis region. Under long-term debt, the No-Lending Condition (5) and the Participation Constraint (6) from Section 3 will still characterize the lower and upper bounds of the Crisis region.

Starting first with the No-Lending Condition, note that expression V^D in equation (5) is the same as equation (8): the value of default when nobody lends does not depend on the type of bond. We now compute V^R in the NLC (5). Recall that under $\delta = 0$ (the short-term debt case), the government paid its total outstanding obligations B in the current period and financed g with y thereafter. This was equivalent to a situation where lenders froze credit for one period. The difference when $\delta > 0$ is that now the government only repays δB in the current period, and what happens thereafter depends on the duration of the credit

[5]From the previous example for $T = 2$, $q = \beta/(1 - \beta\delta)$ also holds when $B' \in [\underline{B}, \bar{B}]$ and $t = 1$. Therefore, $q = \beta/(1 - \beta\delta)$ is valid for the general T case when $t = T - 1$.

market freeze.

If we consider the extreme case where the credit market freezes loans for infinitely many periods, the government would never be able to issue debt again. Claim 16 in Section $B.2.1$ of the Appendix shows that the value function V^R that results after iterating the law of motion for i_t (18) is

$$V^R(B_0,\delta)=\sum_{t=0}^{\infty}\beta^t u(g_t) \quad s.t.\ g_t=y-\delta^t B_0. \tag{22}$$

Expression (22) implicitly states that creditors never lend even when the government honours its obligations in every period. Since this is a strong requirement, we implement the more realistic assumption that lenders freeze loans only for an arbitrary number of periods. In this set-up, we fix the credit freeze to one period, what allows us to compare our results with Cole and Kehoe (2000). Hence, the government value function yields

$$V^R(B_0,\delta)=u(y-B_0)+\beta\frac{u\left(y-\frac{1-\beta}{1-\beta\delta}\delta B_0\right)}{1-\beta}, \tag{23}$$

and the final expression for the No-Lending Condition under a one-period market freeze is

$$\frac{u(Zy)}{1-\beta}>u(y-B_0)+\beta\frac{u\left(y-\frac{1-\beta}{1-\beta\delta}\delta B_0\right)}{1-\beta}. \tag{24}$$

Expression V^D in the Participation Constraint (6) only needs substituting the stationary price q by $\hat{\beta}/\left(1-\hat{\beta}\delta\right)$ in eq. (7). For V^R, we proceed with a backward induction logic similar to the one-period bond case, but using budget constraint (17) and adding the law of motion of issuances (18). Therefore, if no default occurred after the T periods of transition, the government problem at the No-Default region is

$$\max_{\substack{\{g_s\}_{s=T}^{\infty},\{i_s\}_{s=T}^{\infty},\\ \{B_{s+1}\}_{s=T}^{\infty}}}\sum_{s=T}^{\infty}\beta^s u(g_s) \tag{25}$$

$$\begin{aligned} s.t.\quad & g_s+B_s=y+q_s i_s \quad \forall s\geq T\\ & i_s=B_{s+1}-\delta B_s \quad \forall s\geq T\\ & q_s=\beta/(1-\beta\delta) \quad \forall s\geq T\\ & B_T=\underline{B} \quad given\\ & B_s\leq\underline{B} \quad \forall s\geq T. \end{aligned}$$

(Refer to Section $B.3.1$ in the Appendix for the full solution.) The optimality condition $u'(g_t) = u'(g_{t+1})$ characterizes a constant government spending given by

$$\bar{g} = y - \left(\frac{1-\beta}{1-\beta\delta}\right)\underline{B}.$$

This last expression leads to the continuation payoff $u(\bar{g})/(1-\beta)$ once the government reaches the No-Default region. Using this result when the government problem starts with an initial debt $B_0 > \underline{B}(\delta)$ and debt is lowered in T periods, gives[6]

$$V^T(B_0,\delta) = \max_{\substack{\{g_t\}_{t=0}^{T-1},\ \{i_t\}_{t=0}^{T-1},\\ \{B_{t+1}\}_{t=0}^{T-1}}} \sum_{t=0}^{T-1} \hat{\beta}^t u(g_t) + \hat{\beta}^{T-1}\beta \frac{u\left(y - \left(\frac{1-\beta}{1-\beta\delta}\right)\underline{B}\right)}{1-\beta} + \left[\frac{1-\hat{\beta}^{T-1}}{1-\hat{\beta}}\right]\beta\pi\frac{u(Zy)}{1-\beta} \tag{26}$$

$$\begin{aligned} s.t.\quad & g_t + B_t = y + q_t^T i_t \quad \forall t = 0,1,...,T-1 \\ & i_t = B_{t+1} - B_t\delta \quad \forall t = 1,...,T-2 \\ & i_t = \underline{B} - B_t\delta \quad t = T-1 \\ & q_t^T = \beta E_t\left[z_{t+1}\left(1 + q_{t+1}^T\delta\right)\right] \\ & z_{-1} = 1, \quad B_0, \quad \underline{B} \quad given, \end{aligned}$$

with associated Euler Equation

$$u'(g_t) = u'(g_{t+1})\left[\frac{\hat{\beta}\left(1+\delta q_{t+1}^T\right)}{q_t^T}\right]$$

Given the assumptions on the international credit market, it is immediate to see that the term in square brackets in the Euler Equation is equal to 1. Particularly, creditors' indifference condition requires balancing today's cost of lending with tomorrow's benefit of being paid a coupon *plus* the possibility of reselling the bond. As a result, government spending will be constant during the first $t = 0,1,...,T-1$ periods as in the short-term debt case.

Using again the T budget constraints, we can obtain the expression for the optimal $g^T(B_0,\delta)$ when the government lowers debt in T periods to exit the Crisis region – for a full derivation, see Section $B.3.3$ in the Appendix.

$$g^T(B_0,\delta) = y - \left(1 + q_0^T\delta\right)\left[B_0 - \underline{B}\prod_{j=0}^{T-1}\frac{q_j^T}{1+q_j^T\delta}\right]\left[1 + \sum_{k=0}^{T-2}\frac{\prod_{j=0}^{T-2-k} q_j^T}{\prod_{j=1}^{T-1-k}\left(1+q_j^T\delta\right)}\right]^{-1}$$

[6] The solution to this problem is in Section $B.3.2$ of the Appendix.

With these results, we define the Participation Constraint (6) as

$$\max\left\{V^{1}\left(B,\delta\right),V^{2}\left(B,\delta\right),...,V^{\infty}\left(B,\delta\right)\right\}\geq u\left(Zy+\frac{\hat{\beta}}{1-\hat{\beta}\delta}B\right)+\beta\frac{u\left(Zy\right)}{1-\beta},\tag{27}$$

where

$$V^{\infty}\left(B,\delta\right)=\frac{u\left(y-\left[\frac{1-\hat{\beta}}{1-\hat{\beta}\delta}\right]B\right)}{1-\hat{\beta}}+\frac{\beta\pi u\left(Zy\right)}{\left(1-\hat{\beta}\right)\left(1-\beta\right)}.$$

Finally, the equality of (27) solves $\bar{B}\left(\pi,\delta\right)$ – which we denote as $\bar{B}\left(\delta\right)$.

5.2 Zero- and positive-probability crisis equilibrium

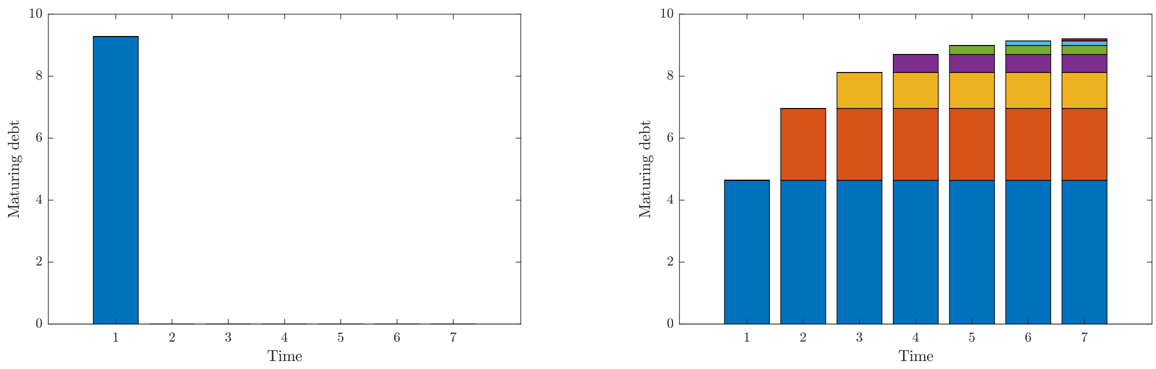
We numerically show that a nonempty Crisis region can exist using long-term debt for $\pi\geq 0$. Following this, we illustrate different debt trajectories $\{B_{t+1}\}$ under alternative plans of lowering debt in T periods. All these proofs are relegated to the Appendix.

Lemma 17 shows that there is one $\underline{B}\left(\delta\right)$ defining a lower bound to the Crisis region when $\delta>0$. For the upper bound $\bar{B}\left(\pi,\delta\right)$, Lemma 19 argues that $V^{\infty}\left(B,\delta\right)$ dominates any other policy $T<\infty$ when π goes to zero in the PC. Intuitively, when creditors expect default to occur with zero probability in the Crisis region, the government has no incentive to lower debt. Thus, the Participation Constraint (27) becomes

$$\frac{u\left(y-\left[\frac{1-\beta}{1-\beta\delta}\right]\bar{B}\left(0,\delta\right)\right)}{1-\beta}=u\left(Zy+\frac{\beta}{1-\beta\delta}\bar{B}\left(0,\delta\right)\right)+\beta\frac{u\left(Zy\right)}{1-\beta}$$

In the Appendix, Lemma 20 proves that there exists a unique level of debt $\bar{B}\left(0,\delta\right)$ that equates the previous expression – Lemma 21 does the same for $\bar{B}\left(\pi,\delta\right)$ at strictly positive probabilities of default.

Before proceeding to the characterization of a nonempty Crisis region, it is worth noting the difference between *the amount of maturing debt* versus *the total amount of debt.* To understand this, we denote $\hat{B}\equiv B/\left(1-\delta\right)$ the total outstanding obligations for a given debt level B. In the model with one-period bonds of the previous sections, any bond issued in period t matured at $t+1$. As a result, the amount of debt maturing at $t+1$ equated total outstanding debt, namely, $B=\hat{B}$. With long-term debt, however, the amount of maturing debt in period $t+1$ only pays a fraction of total outstanding debt $\hat{B}$. This is because coupons divide the payment of $\hat{B}$ into more periods, thus decreasing maturing debt for every t. The plot below highlights the different repayment schedules when the government repays the same stock of debt $\hat{B}$ using either one-period bonds (Figure 5a) or long-term bonds with



Panel 5a

Panel 5b

Fig. 5. Maturing debt accumulated over periods, long-term debt. Panel 5a: $\delta = 0$; Panel 5b: $\delta = 0.5$; total outstanding debt $\simeq 9.28$.

infinite coupon payments (Figure 5*b*).

Figure 5*b* shows how coupon payments generate the same total debt as in Figure 5*a* by allocating portions of debt more evenly across periods. This lowers the amount of maturing debt at every t, thus allowing the government to stand a higher total amount of debt $\hat{B}$. As a result of this, the government's default incentives should intuitively decrease as δ rises, thus increasing thresholds $\underline{B}(\delta)$ and $\bar{B}(\delta)$ where the government is indifferent between repayment or default.

Using the previous choice of parameters, Figure 6 characterizes three nonempty Crisis regions. To obtain this graph, we first replace the level of debt in the No-Lending Condition (24) and in the Participation Constraint (27) by $\hat{B}(1-\delta)$. We do this to derive conclusions on the thresholds $\underline{B}(\delta)$ and $\bar{B}(\delta)$ based on total amount of debt, when comparing them under short-term ($\delta = 0$) and long-term debt ($\delta > 0$). Then, we define

$$
\begin{aligned}
U_{NLC}(B;\delta) &\equiv \frac{u(Zy)}{1-\beta} - u(y-(1-\delta)B) - \beta\frac{u\left(y - \frac{1-\beta}{1-\beta\delta}\delta(1-\delta)B\right)}{1-\beta} \\
U_{PC}^{\pi>0}(B;\delta) &\equiv \frac{u\left(y - \frac{1-\hat{\beta}}{1-\hat{\beta}\delta}(1-\delta)B\right)}{1-\hat{\beta}} + \frac{\beta\pi}{1-\hat{\beta}}\frac{u(Zy)}{1-\beta} - u\left(Zy + \frac{\hat{\beta}}{1-\hat{\beta}\delta}(1-\delta)B\right) - \beta\frac{u(Zy)}{1-\beta}.
\end{aligned}
$$

and plot these curves as δ varies – Section $B.3.5$ of the Appendix analytically describes their behavior as B increases. Debt levels B satisfying $U_{PC}^{\pi>0}(B;\delta) \geq 0 \geq U_{NLC}(B;\delta)$ when $\delta = \{0, 0.1, 0.2\}$ characterize intervals where self-fulfilling debt crises can occur with positive probability. When $\delta = 0$, the government repays its total outstanding obligations in one period. Instead, when $\delta > 0$, total debt is spread across infinitely many periods, and from our previous intuition this should allow $\hat{B}$ to be higher. In particular, some debt levels that initially violated eqs. (24) and (27) when $\delta = 0$ will now satisfy them since a higher δ

decreases default incentives. Ultimately, this implies that the bounds of the Crisis region can only increase in order to keep the government indifferent between defaulting and repaying. The next Proposition formalizes this result.

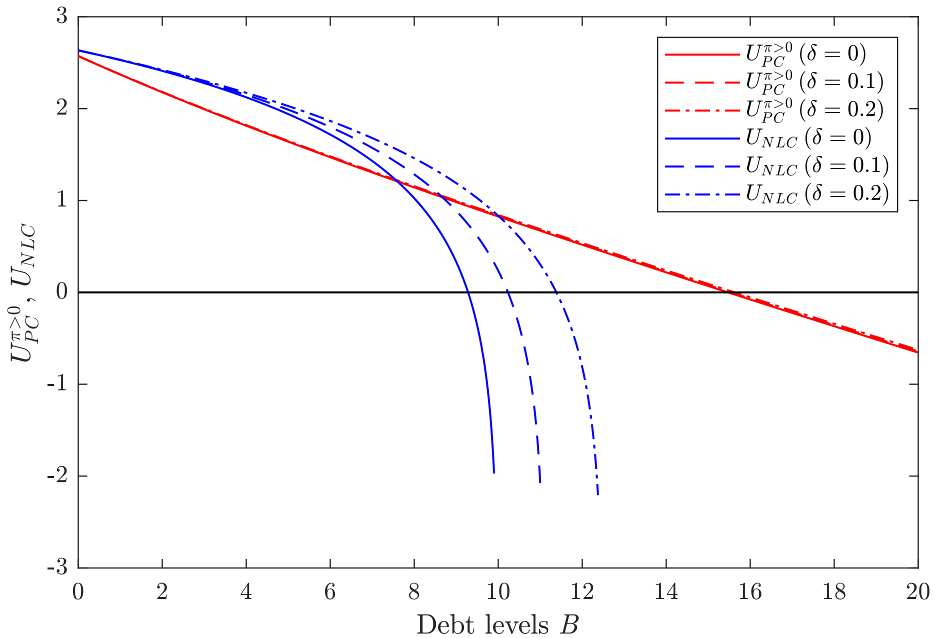


Fig. 6. Crisis region when $\pi > 0$ at different debt maturities δ. ($\pi = 0.0001$, $\beta = 0.96$, $\delta = \{0, 0.1, 0.2\}$, $Z = 0.9$, $y = 10$ and $u(\cdot) = \ln(\cdot)$.)

Proposition 5 *When δ rises, the bounds $\underline{B}(\delta)$ and $\bar{B}(\pi, \delta)$ of the Crisis region increase.*

Proof. See Appendix. ∎

Figure 5*b* illustrated how an increase in debt maturity made future coupon payments a larger share of total debt. In line with this, the result in Proposition 5 analytically states that higher debt maturity decreases government's incentives to default. When δ increases, the government faces lower obligations at early periods, therefore increasing government's incentives to (i) repay when nobody lends (i.e., satisfy the NLC (24)), and (ii) remain in the international credit market (i.e., satisfy the PC (27)).

Figure 7 plots trajectories of debt when $\delta > 0$ and the government chooses the optimal time to exit the Crisis region by lowering the total amount of debt. It is straightforward to observe that the optimal government policies $\{B_{t+1}\}$ imply trajectories of debt that increase T as B_0 rises. This is similar to the short-term debt case. The bounds of the Crisis region when $\delta > 0$, however, are above those of short-term debt – what coincides with the finding in Proposition 5. When $\delta > 0$, the government can tolerate a higher total amount of debt since coupon payments allocate a larger share of obligations in future periods.

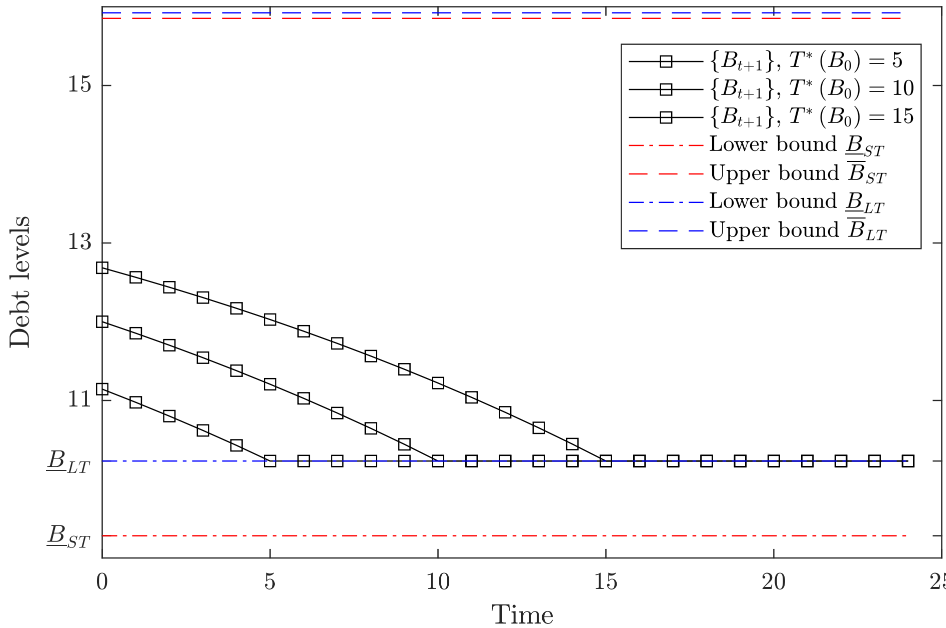


Fig. 7. Optimal $\{B_{t+1}\}$ in the Crisis region $\left[\underline{B}, \overline{B}\right]$, at different B_0 and associated $T^*(B_0$ long-term debt. ($\pi = 0.0001$, $\beta = 0.96$, $\delta = 0.1$, $Z = 0.9$, $y = 10$ and $u(\cdot) = \ln(\cdot)$.)

6 Welfare analysis

In this section, we analyze the welfare implications of lengthening debt maturity. We first explore the impact of short-term and long-term debt in the government's optimal debt choices. We then turn to the numerical comparison of welfare under bonds of different maturity.

Starting at the same level of total outstanding debt, Figure 8 illustrates the optimal path $\{B_{t+1}\}$ under the two classes of bonds. In particular, we set $\pi = 0.0001$, and parameters $\delta = 0$ and $\delta = 0.1$ denote one-period and long-term bonds, respectively. The plot shows that an initial debt level $\hat{B} \approx 11.15$ is closer to the lower bound of the Crisis region under long-term debt ($\underline{B}(0.1) \approx 10.23$) than under short-term debt ($\underline{B}(0) \approx 9.28$). The reason why $\underline{B}(0.1) > \underline{B}(0)$ is that creditors now expect default to occur at larger debt levels when δ is higher. Recall that when debt maturity increases, some debt levels above threshold $\underline{B}(0)$ will now be independent from the sunspot variable ζ_t. At those levels of debt, not only the government's incentives to default decreased, but also creditors know that the government can tolerate a larger stock of total debt. As a result, the government now repays these debt levels with probability 1, until it reaches the new upper bound $\underline{B}(0.1)$.[7]

Moreover, the number of periods that the government takes to escape the Crisis region $T^*(B_0)$ also varies as δ increases. The plot shows that when $\delta = 0$, the government avoids

[7]Proposition 5 of the previous section analytically proved this result.

self-fulfilling debt crises by exiting the Crisis region in $T(\hat{B}) = 10$ periods. When $\delta = 0.1$, however, the government chooses a faster policy $T(\hat{B}) = 5$. Specifically, when δ increases, any total debt is now closer to the new lower bound of the No-Default region. This proximity of $\hat{B}$ to $\underline{B}(0.1)$ provides incentives to decrease $T(\hat{B})$, and the government therefore becomes less vulnerable to self-fulfilling debt crises.

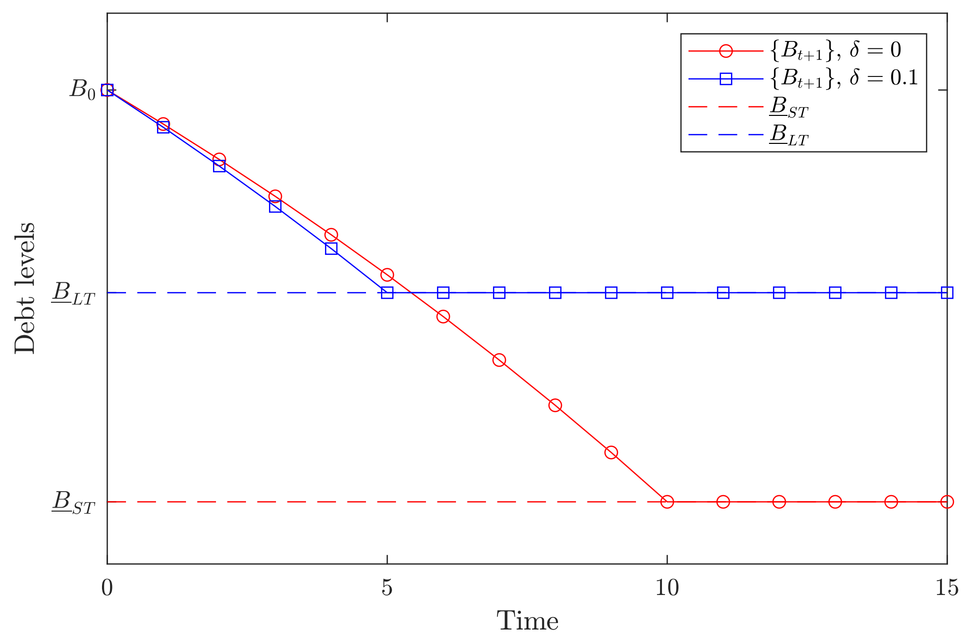


Fig. 8. Optimal $\{B_{t+1}\}$ for short-term and long-term bonds, at the same B_0. ($\pi = 0.0001$, $\beta = 0.96$, $\delta = \{0, 0.1\}$, $Z = 0.9$, $y = 10$ and $u(\cdot) = \ln(\cdot)$.)

Table 1

Welfare under short-term and long-term debt

	$T(\hat{B})$	$V(\hat{B}, T(\hat{B}), \delta)$
Short-term debt ($\delta = 0$)	10	56.4204
Long-term debt ($\delta = 0.1$)	5	56.4272

Note: utility and parameter values as reported in text.

Table 1 summarizes some important results from this example. Furthermore, the second column reports a greater welfare for bonds of longer maturity. Motivated by this observation, we numerically explore the welfare implications of bond maturity for a wider range of initial debt levels. First, we build the value functions for short-term and long-term debt at different T's. Second, we evaluate each value function for every total outstanding debt $\hat{B}$ that lies in the Crisis region. Third, for each $\hat{B}$, we identify which policy T achieves the highest value $V^T(\cdot)$. We repeat this process both for short-term and long-term debt. Fourth, we build the envelope function that collects the highest $V^T(\cdot)$, for every $\hat{B}$ and for each separate case, $\delta = 0$ and $\delta > 0$. Specifically, for each $\delta = \{0, 0.1\}$, the envelopes will show the value $V^T(\cdot)$, for each $\hat{B}$ in the Crisis region and at the optimal $T(\hat{B})$. Finally, we use these envelopes to perform welfare comparisons.

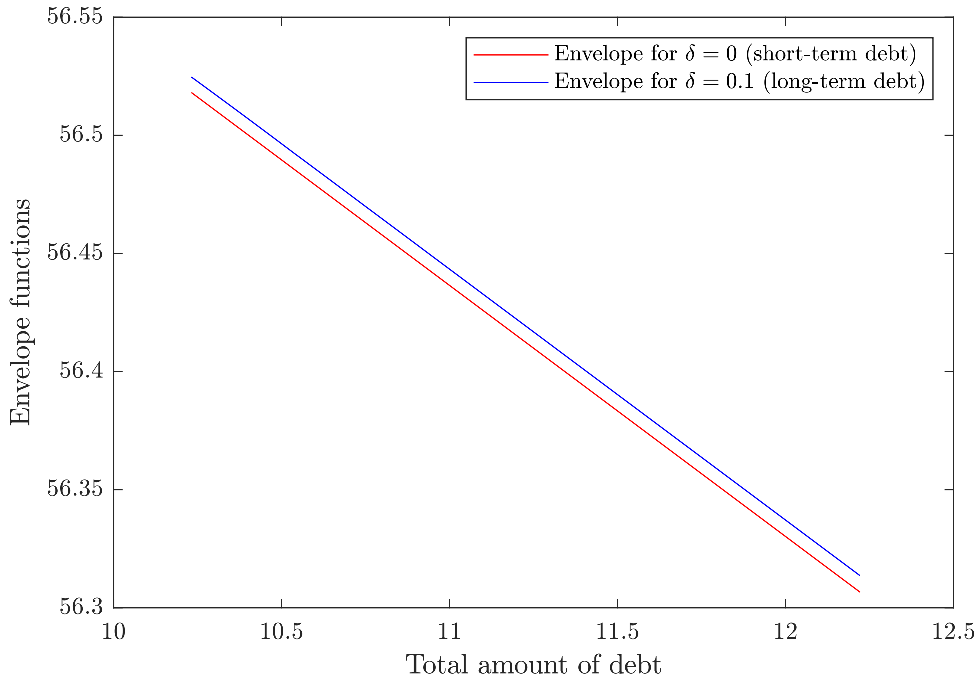


Fig. 9. Envelope functions for short-term and long-term debt, as total amount of debt varies. ($\pi = 0.0001$, $\beta = 0.96$, $\delta = \{0, 0.1\}$, $Z = 0.9$, $y = 10$ and $u(\cdot) = \ln(\cdot)$.)

Figure 9 shows the envelope functions under short-term (red line) and long-term debt, for $\delta = 0.1$ and 0.2 (blue line and dark line, respectively).[8] It is immediate to observe that as δ increases, the envelopes are strictly above the one under short-term debt, for any initial total debt in the Crisis region. In other words, the government attains a higher total utility at any initial $\hat{B}$ when debt maturity increases. To understand this, recall that when $\delta = 0$ the government paid back total debt in one period. When $\delta > 0$, however, the government repays the same total amount of debt using coupons in each period. Precisely, a higher δ decreases the value of the first coupons by transferring a portion of total debt to the future. Since this decreases both default incentives *and* $T(B_0)$, the likelihood of self-fulfilling debt crises declines. This is a first gain coming from a higher δ. But now, discounting also generates an additional government gain coming from a shorter transition $T(B_0)$. In particular, when the optimal $T(B_0)$ decreases, the government reaches the No-

[8]This figure illustrates the government payoffs for a portion of total debt levels B. Refer to Section $B.4.2$ of the Appendix for a detailed description of the government's payoff for debt levels starting at $B = 0$. Figures 12 and 13 in Section $B.4.2$ of the Appendix also illustrate how welfare increases when π rises.

Default region in a smaller number of periods, and therefore it discounts less the payoff $u(\bar{g})$ attained when the government arrives to that region. These gains finally explain why welfare increases compared to the one-period bonds case.

7 Conclusion

This paper explored the impact of debt maturity in a model of self-fulfilling debt crises, featuring a small open economy where a benevolent government maximized household's utility and could default on its debt. The main finding showed that switching from short-term debt to long-term debt can improve the welfare of the economy. Specifically, the first part of the paper characterized an interval of debt levels for short-term debt, where creditors' expectations of default could trigger government's default with a fixed probability. The second part of the paper modified the initial set-up by switching to long-term debt in the form of a coupon-paying bond. This type of bond changed the structure of the equilibrium price of debt. Furthermore, long-term debt shifted upward the bounds of the interval of debt levels where crises are belief-driven. Finally, the paper numerically showed that long-term bonds can improve the economy's welfare by decreasing default incentives of the government and the likelihood of self-fulfilling debt crises.

This paper suggested that lengthening debt maturity can increase welfare using an economic environment as in Cole and Kehoe (2000). Other works in the literature, however, have documented trade-offs coming from the use of both short-term and long-term debt in models of sovereign default. For example, Arellano and Ramanarayanan (2012) explore hedging and incentive benefits for the government that are linked to the issuance of debt of different maturities. Whether there is an optimal combination of short-term and long-term bonds within a model of self-fulfilling debt crises remains an open question. We plan to explore this question by incorporating the possibility to choose between different debt instruments in our analysis in future work.

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Appendix A - Short-term debt

7.1 Determination of the equilibrium price

When B' lies in the Crisis region, creditors expect the government to default with probability π. Thus, the equilibrium price equation (3) leads to

$$q\left(s, B'\right) = \beta\left[\pi(0) + (1-\pi)\right] \Rightarrow q\left(s, B'\right) = \beta(1-\pi)$$

If B' lies in the Default region, the international credit market expects the government to default with probability 1, and therefore $q(s, B') = 0$. In contrast, if B' lies in the No-Default region, the government never defaults and $q(s, B') = \beta$.

7.2 Determination of the lower bound of the Crisis region

To show that there exists a unique $B = \underline{B}$ solution to equation (10) at equality, the next result follows.

Lemma 6 *There exists a unique $\underline{B}$ that is solution to*

$$\frac{u(Zy)}{1-\beta} = u(y - \underline{B}) + \beta\left[\frac{u(y)}{1-\beta}\right].$$

Proof. Using the No-Lending Condition, let

$$F(B) = \frac{u(Zy)}{1-\beta} - u(y-B) - \beta\frac{u(y)}{1-\beta} = 0.$$

Since (i) $F(0) = [u(Zy) - u(y)]/(1-\beta) < 0$, (ii) $F(B \to y) = u(Zy)/(1-\beta) - u(0) - \beta u(y)/(1-\beta) \to \infty$, and (iii) $dF(B)/dB = u'(y-B) > 0$, then there exists a unique $\underline{B} > 0$ solution to $F(B) = 0$.
■

7.3 Determination of the upper bound of the Crisis region

7.3.1 Optimal policy in the No-Default region under short-term debt

Claim 7 solves the government maximization problem under short-term debt. In particular, it shows that the optimal paths for government spending $\{g_t\}$ and bonds $\{B_{t+1}\}$ are constant when there is no risk of default (i.e., $\pi = 0$).

Claim 7 (Short-term debt) *If $\pi = 0$ and given an initial debt $B_{t'} = B \leq \underline{B}$, then government spending is constant and equal to*

$$g = y - B(1-\beta),$$

and government debt B_{s+1} equals the initial debt $B_{t'}$ for $s \geq t'$.

Proof. Starting at t' and for any initial $B_{t'} = B$, a general version of the government problem (equation (11) in the main text) can be written as

$$\max_{\substack{\{B_{s+1}\}_{s=t'}^{\infty} \\ \{g_s\}_{s=t'}^{\infty}}} \sum_{s=t'}^{\infty} \beta^s u(g_s) \tag{28}$$

$$
\begin{aligned}
s.t. \quad & g_s + B_s = y + q_s B_{s+1} \quad \forall s \geq t' \\
& B_{t'} = B \text{ given, and } B \leq \underline{B} \\
& B_s \leq \underline{B} \quad \forall s \geq t'.
\end{aligned}
$$

When $\pi = 0$, the price q of a riskless bond equals β and the government budget constraint is

$$g_s + B_s = y + \beta B_{s+1} \quad \forall s \geq t'.$$

The optimal paths $\{g_s\}_{s=t'}^{\infty}$, $\{B_{s+1}\}_{s=t'}^{\infty}$ of the government maximization problem (28) solve

$$\mathcal{L} = \sum_{s=t'}^{\infty} \{\beta^s u(g_s) - \lambda_s [g_s + y + \beta B_{s+1} - B_s]\}$$

The first-order and the transversality conditions are

$$
\begin{array}{ccccccc}
\{g_s\} & : & \beta^s u'(g_s) - \lambda_s = 0 & \Rightarrow & \beta^s u'(g_s) & = & \lambda_s \\
\{B_{s+1}\} & : & -\beta\lambda_s - \lambda_{s+1}(-1) = 0 & \Rightarrow & \beta\lambda_s & = & \lambda_{s+1}
\end{array}
$$

$$\lim_{s\to\infty} \beta^s \lambda_s B_{s+1} = 0.$$

From the first-order conditions,

$$u'(g_s) = u'(g_{s+1}),$$

and this implies a constant government spending at $g_s = g_{s+1} = g$.

We now show that B_s equals the initial $B_{t'} = B$ for every $s \geq t'$. Plugging g in two consecutive budget constraints, the difference yields

$$B_{s+2} - B_{s+1} = \frac{1}{\beta}(B_{s+1} - B_s).$$

Let $\Delta_{s+1} = B_{s+1} - B_s$ and notice that $\beta^{-1} > 1$. When $\Delta_{s+1} < 0$, the optimal path of debt diverges to negative infinity. In this case, the government can always do better by increasing g_t but this violates the sign of the transversality condition. When $\Delta_{s+1} > 0$, government debt diverges to positive infinity. But this now violates constraint $B_s \leq \underline{B}$ since the optimal path $\{B_{s+1}\}_{s=t'}^{\infty}$ will hit the lower bound of the Crisis region for some s. Moreover, this violates the assumption that lenders have a finite wealth. Thus, the only Δ_{s+1} consistent with the transversality condition and with optimality of government spending is $\Delta_{s+1} = 0$, namely, $B_{s+1} = B_s$. Since this holds for every $s \geq t'$, the optimal path $\{B_{s+1}\}_{s=t'}^{\infty}$ is constant and equal to $B_{t'} = B$, also satisfying constraint $B_s \leq \underline{B}$ for every $s \geq t'$.

Finally, optimal government spending in terms of the initial debt level $B_{t'} = B$ yields

$$g + B = y + \beta B \Rightarrow g = y - B(1 - \beta)$$

■

7.3.2 General problem when debt is lowered in T periods

Be $\bar{g} = y - (1-\beta)\underline{B}$. When $T = \{1, 2, 3\}$, the government's objective functions are

$$u(g_0) + \beta \frac{u(\bar{g})}{1-\beta}$$
$$u(g_0) + \hat{\beta} u(g_1) + \hat{\beta}\beta \frac{u(\bar{g})}{1-\beta} + \beta\pi \frac{u(Zy)}{1-\beta}$$
$$u(g_0) + \hat{\beta} u(g_1) + \hat{\beta}^2 u(g_2) + \hat{\beta}^2 \beta \frac{u(\bar{g})}{1-\beta} + \beta\pi \frac{u(Zy)}{1-\beta}\left(1+\hat{\beta}\right)$$

respectively. Therefore, the objective function of the government over T periods is

$$\sum_{k=0}^{T-1} \hat{\beta}^k u(g_k) + \hat{\beta}^{T-1} \beta \frac{u(\bar{g})}{1-\beta} + \beta\pi \frac{u(Zy)}{1-\beta} \sum_{k=0}^{T-2} \hat{\beta}^k,$$

or, alternatively,

$$\sum_{k=0}^{T-1} \hat{\beta}^k u(g_k) + \hat{\beta}^{T-1} \beta \frac{u(\bar{g})}{1-\beta} + \left[\frac{1-\hat{\beta}^{T-1}}{1-\hat{\beta}}\right] \beta\pi \frac{u(Zy)}{1-\beta}.$$

7.3.3 Solution of the general problem for any $T \geq 2$

The general problem is

$$V^T(B_0) = \max_{\substack{\{B_{t+1}\}_{t=0}^{T-1}, \\ \{g_t\}_{t=0}^{T-1}}} u(g_0) + \hat{\beta} u(g_1) + \hat{\beta}^2 u(g_2) + \ldots + \hat{\beta}^{T-1} u(g_{T-1}) + \left[\frac{1-\hat{\beta}^{T-1}}{1-\hat{\beta}}\right] \beta\pi \frac{u(Zy)}{1-\beta} + \hat{\beta}^{T-1} \beta \frac{u(\bar{g})}{1-\beta}$$

$$\begin{aligned} s.t. \quad & g_t = y - B_t + q_t B_{t+1} \quad \forall t = 0, 1, .., T-2 \\ & g_t = y - B_t + q_t \underline{B} \quad t = T-1 \\ & q_t = \beta E[z_{t+1}], \quad z_t = 1 \\ & B_0, \quad \underline{B} \quad given; \quad \bar{g} = y - (1-\beta)\underline{B}. \end{aligned}$$

If B_0 lies in the Crisis region, then $q_t^T = \hat{\beta}$. After plugging the constraints, the first-order conditions for t yield

$$\{B_t\} : \hat{\beta}^{t-1} u'(g_{t-1}) \hat{\beta} + \hat{\beta}^t u(g_t)(-1) = 0 \Rightarrow u'(g_{t-1}) = u'(g_t) \Rightarrow g_{t-1} = g_t$$

From the first-order condition, $g_t = g_{t+1}$ for every $t = 0, 1, ..., T-1$. Furthermore, since g_0 depends on the initial debt level B_0, then we label constant government spending as $g_t = g^T(B_0)$ for every $t = 0, 1, ..., T-1$.

7.3.4 Expression for $g^T(B_0)$

To derive an expression for $g^T(B_0)$, we first replace government spending by $g^T(B_0)$ in the T budget constraints of the previous problem. Then, we multiply each expression by $\hat{\beta}^t$ with $t = 0, 1, ..., T-1$, to obtain

$$\begin{aligned}
\hat{\beta}^{0} g^{T}\left(B_{0}\right)+\hat{\beta}^{0} B_{0} &= \hat{\beta}^{0} y+\hat{\beta}^{0} \hat{\beta} B_{1} \\
\hat{\beta}^{1} g^{T}\left(B_{0}\right)+\hat{\beta}^{1} B_{1} &= \hat{\beta}^{1} y+\hat{\beta}^{1} \hat{\beta} B_{1} \\
&\vdots \\
\hat{\beta}^{T-2} g^{T}\left(B_{0}\right)+\hat{\beta}^{T-2} B_{T-2} &= \hat{\beta}^{T-2} y+\hat{\beta}^{T-2} \hat{\beta} B_{T-1} \\
\hat{\beta}^{T-1} g^{T}\left(B_{0}\right)+\hat{\beta}^{T-1} B_{T-1} &= \hat{\beta}^{T-1} y+\hat{\beta}^{T-1} \beta \underline{B}.
\end{aligned}$$

Adding up these terms leads to

$$g^{T}\left(B_{0}\right) \sum_{k=0}^{T-1} \hat{\beta}^{k}+B_{0}=y \sum_{k=0}^{T-1} \hat{\beta}^{k}+\hat{\beta}^{T-1} \beta \underline{B},$$

and solving for $g^{T}\left(B_{0}\right)$ renders

$$g^{T}\left(B_{0}\right)=y-\left[\frac{1-\hat{\beta}}{1-\hat{\beta}^{T}}\right]\left[B_{0}-\hat{\beta}^{T-1} \beta \underline{B}\right].$$

7.3.5 The upper bound of the Crisis region when $\pi=0$

Before characterizing the existence and uniqueness of the upper bound $\bar{B}$ of the Crisis region, we introduce the next result that holds when $\pi=0$. When the probability of defaults is null, the government should always choose to roll-over its current level of debt instead of adopting any policy of lowering debt in T periods.

Lemma 8 *Suppose that $\pi \rightarrow 0$ and consider $V^{\infty}\left(B_{0}\right)-V^{T}\left(B_{0}\right)$. Then,*

$$V^{\infty}\left(B_{0}\right)-V^{T}\left(B_{0}\right)>0$$

for every $T<\infty$.

Proof. When $\pi \rightarrow 0$, the value function at $T=1$ is defined as

$$V^{1}\left(B_{0}\right)=u\left(y-B_{0}+\beta \underline{B}\right)+\beta \frac{u\left(y-(1-\beta) \underline{B}\right)}{1-\beta},$$

while for $T \geq 2$,

$$V^{T}\left(B_{0}\right)=\left[\frac{1-\beta^{T}}{1-\beta}\right] u\left(y-\left[\frac{1-\beta}{1-\beta^{T}}\right] B_{0}+\left[\frac{1-\beta}{1-\beta^{T}}\right] \beta^{T} \underline{B}\right)+\beta^{T} \frac{u\left(y-(1-\beta) \underline{B}\right)}{1-\beta}.$$

Computing $V^{\infty}\left(B_{0}\right)-V^{1}\left(B_{0}\right)$ renders

$$\frac{1}{1-\beta}\left[u\left(y-(1-\beta) B_{0}\right)-(1-\beta) u\left(y-B_{0}+\beta \underline{B}\right)-\beta u\left(y-(1-\beta) \underline{B}\right)\right],$$

and since

$$(1-\beta)\left(y-B_{0}+\beta \underline{B}\right)+\beta\left(y-(1-\beta) \underline{B}\right)=y-(1-\beta) B_{0},$$

then by concavity of $u(\cdot)$ and Jensen's inequality, $u(y-(1-\beta)B_0)-(1-\beta)u(y-B_0+\beta\underline{B})-\beta u(y-(1-\beta)\underline{B})>0$. Thus, $V^{\infty}(B_0)-V^{1}(B_0)>0$.

With a similar argument for $V^{\infty}(B_0)-V^{T}(B_0)$ when $T\geq 2$,

$$V^{\infty}(B_0)-V^{T}(B_0)=\frac{1}{1-\beta}\left[u(y-(1-\beta)B_0)-\left(1-\beta^{T}\right)u\left(g^{T}(B_0)\right)-\beta^{T}u(y-(1-\beta)\underline{B})\right],$$

and since

$$\left(1-\beta^{T}\right)\left[y-\left[\frac{1-\beta}{1-\beta^{T}}\right]B_0+\left[\frac{1-\beta}{1-\beta^{T}}\right]\beta^{T}\underline{B}\right]+\beta^{T}\left[y-(1-\beta)\underline{B}\right]=y-(1-\beta)B_0,$$

then, $V^{\infty}(B_0)-V^{T}(B_0)>0$ – using concavity of $u(\cdot)$ and Jensen's inequality. ■

Having proved that $V^{\infty}>V^{T}$ for $\pi>0$ and $T<\infty$, we replace the left-hand side of equation (15) by $V^{\infty}(B)$ and proceed to characterize $\bar{B}(0)$ (which for simplicity we call $\bar{B}$).

Lemma 9 *There exists a unique $\bar{B}$ that solves*

$$\frac{u\left(y-(1-\beta)\bar{B}\right)}{1-\beta}=u\left(Zy+\beta\bar{B}\right)+\beta\frac{u(Zy)}{1-\beta}$$

Proof. Using the Participation Constraint with $V^{\infty}(B)$ in the right-hand-side, we can define the following function

$$H(B;\beta,y,Z)=\frac{u(y-(1-\beta)B)}{1-\beta}-u(Zy+\beta B)-\beta\frac{u(Zy)}{1-\beta}=0$$

Given that (i) $H(0;\cdot)=[u(y)-u(Zy)]/(1-\beta)>0$, (ii) $H(B\rightarrow y/(1-\beta);\cdot)=u(0)/(1-\beta)-u(Zy+\beta y/(1-\beta))-\beta u(Zy)/(1-\beta)\rightarrow-\infty$, and (iii) $\partial H(B;\cdot)/\partial B=-u'(y-(1-\beta)B)-u'(Zy+\beta B)\beta<0$, then there exists a unique $\bar{B}>0$ such that the Participation Constraint is satisfied with equality. Moreover, for any $B<\bar{B}(0)$, then $V^{\infty}(B)>V^{D}(B)$, and viceversa. ■

7.3.6 Figure 2

Following a logic similar to Cole and Kehoe (2000), we describe the behavior of equations (10) and (15). Rewriting first

$$U_{PC}^{\pi=0}(B)\equiv\frac{u(y-(1-\beta)B)}{1-\beta}-u(Zy+\beta B)-\beta\frac{u(Zy)}{1-\beta}$$

$$U_{NLC}(B)\equiv u(y-B)+\beta\left[\frac{u(y)}{1-\beta}\right]-\frac{u(Zy)}{1-\beta},$$

then it is immediate to observe that they decrease monotonically as B increases, since (i) $U_{PC}^{\pi=0}(B=0)=U_{NLC}(B=0)=[u(y)-u(Zy)]/(1-\beta)>0$, (ii) $\lim_{B\rightarrow y/(1-\beta)}U_{PC}^{\pi=0}(B)=u(0)/(1-\beta)-u(Zy+y\beta/(1-\beta))-\beta u(Zy)/(1-\beta)\rightarrow-\infty$ and $\lim_{B\rightarrow y}U_{NLC}(B)=u(0)+u(y)\beta/(1-\beta)-u(Zy)/(1-\beta)\rightarrow-\infty$, and (iii) $dU_{PC}^{\pi=0}/dB=-u'(y-(1-\beta)B)-u(Zy+\beta B)\beta<0$ and $dU_{NLC}/dB=-u'(y-B)<0$.

7.3.7 The upper bound of the Crisis region when $\pi > 0$

Lemma 10 *For some $\pi > 0$, there exists a unique $\bar{B}$ that solves*

$$\frac{u\left(y - \left(1-\hat{\beta}\right)B_0\right)}{1-\hat{\beta}} + \frac{\beta\pi u(Zy)}{(1-\beta)\left(1-\hat{\beta}\right)} = u\left(Zy + \hat{\beta}\bar{B}\right) + \beta\frac{u(Zy)}{1-\beta}.$$

Proof. As performed in Lemma 9, we use equation (15) to define

$$\tilde{H}(B;\beta,y,Z,\pi) = \frac{u(y-(1-\hat{\beta})B_0)}{1-\hat{\beta}} + \frac{\beta\pi u(Zy)}{(1-\beta)(1-\hat{\beta})} - u\left(Zy+\hat{\beta}\bar{B}\right) - \beta\frac{u(Zy)}{1-\beta} = 0.$$

Given that (i) $\tilde{H}(0;\cdot) = u(y)/(1-\hat{\beta}) - u(Zy)/(1-\beta) + \beta\pi u(Zy)/[(1-\beta)(1-\hat{\beta})] > 0$,[9] (ii) $\tilde{H}(B \to y/(1-\beta);\cdot) = u(0)/(1-\beta) + \beta\pi u(Zy)/[(1-\beta)(1-\hat{\beta})] - u(Zy + \hat{\beta}y/(1-\hat{\beta})) - \beta u(Zy)/(1-\beta) \to -\infty$, and (iii) $d\tilde{H}(B;\cdot)/dB = -u'(y-(1-\hat{\beta})B) - u'(Zy+\beta\hat{B})\hat{\beta} < 0$, then a positive $\bar{B}(\cdot)$ satisfies equation (15) and is unique. ■

7.3.8 Figure 3

Since function $U_{NLC}(B)$ is independent from π, the same result in Section A.3.6 applies. Therefore, defining

$$U_{PC}^{\pi>0}(B) \equiv \frac{u(y-(1-\hat{\beta})B)}{1-\hat{\beta}} + \frac{\beta\pi u(Zy)}{(1-\beta)(1-\hat{\beta})} - u(Zy+\hat{\beta}B) - \beta\frac{u(Zy)}{1-\beta},$$

it is easy to observe that $U_{PC}^{\pi>0}(B)$ monotonically decreases as B increases, since (i) $U_{PC}^{\pi>0}(B=0) = [u(y) - u(Zy)]/(1-\hat{\beta}) + \beta\pi u(Zy)/(1-\beta)(1-\hat{\beta}) > 0$, (ii) $\lim_{B\to y/(1-\beta)} U_{PC}^{\pi>0}(B) = u(0)/(1-\hat{\beta}) + \beta\pi u(Zy)/(1-\beta)(1-\hat{\beta}) - u(Zy + y\hat{\beta}/(1-\beta)) - \beta u(Zy)/(1-\beta) \to -\infty$, and (iii) $dU_{PC}^{\pi>0}(B)/dB = -u(y-(1-\hat{\beta})B) - u(Zy+\hat{\beta}B)\hat{\beta} < 0$.

Lemma 11 *When π increases, the stationary upper bound $\bar{B}(\pi)$ decreases.*

Proof. By Lemma 10 there exists a unique $\bar{B}(\pi)$ that is solution to equation (15) at equality. Rewriting that equation as

$$H = \frac{u\left(y-\left(1-\hat{\beta}\right)B\right)}{1-\hat{\beta}} + \frac{\pi}{\left(1-\hat{\beta}\right)}\frac{\beta u(Zy)}{(1-\beta)} - u\left(Zy+B\hat{\beta}\right) - \frac{\beta u(Zy)}{1-\beta} = 0,$$

we can apply the implicit function theorem to get $\partial B(\pi)/\partial\pi = -(\partial H/\partial\pi)/(\partial H/\partial B)$. Expression $\partial H/\partial B$ and $\partial H/\partial\pi$ yield

$$\frac{\partial H}{\partial B} = -\left[u'\left(y-\left(1-\hat{\beta}\right)B\right) + u'\left(Zy+B\hat{\beta}\right)\hat{\beta}\right] < 0,$$

[9]Since $u(y) > u(Zy)$ and $\lim_{\pi\to 0}\tilde{H}(0;\beta,y,Z,\pi) > 0$, then by continuity of π in $[0,1]$, there is some strictly positive π such that $\tilde{H}(0;\beta,y,Z,\pi)$ is still greater than 0.

$$\frac{\partial H}{\partial \pi} = -\left\{ \beta B \left(\frac{u'\left(y - \left(1-\hat{\beta}\right)B\right)}{1-\hat{\beta}} - u'\left(Zy + B\hat{\beta}\right)\right) + \frac{\beta}{\left(1-\hat{\beta}\right)^2}\left[u\left(y - \left(1-\hat{\beta}\right)B\right) - u\left(Zy\right)\right]\right\}.$$

The term in brackets in the last expression is positive. Also, the No-Lending Condition implies that $y - B < Zy$. Adding $B\hat{\beta}$ to the last inequality and invoking concavity of $u(\cdot)$ renders $u'\left(y - \left(1-\hat{\beta}\right)B\right) > u'\left(Zy + B\hat{\beta}\right)$. Therefore, the first term is positive and $\partial H/\partial\pi < 0$, thus implying that $\partial B(\pi)/\partial\pi < 0$. ■

7.3.9 Proof of Proposition 1

We now present Lemma 12, which will be useful for proving Proposition 2.

Lemma 12 *Suppose that the government wants to run down debt in T periods (i.e., have $B_T = \underline{B}$). Denote the government spending at time t for $t \leq T-1$ along this path as g_t^T. Then,*

1. *optimal g_t^T is constant for all $t \in \{0, ..., T-1\}$, and*
2. *$g_t^T < \bar{g}$, for all $t \in \{0, ..., T-1\}$.*

Proof. Throughout the proof, we assume that B_0 starts in the Crisis region ($B_0 > \underline{B}$), and that the government optimally decides to lower debt in T periods. The government problem becomes

$$\begin{aligned} V^T(B_0) &= \max_{\{B_{t+1}\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \hat{\beta}^t u\left(y - B_t + \hat{\beta}B_{t+1}\right) + \hat{\beta}^{T-1} u\left(y - B_{T-1} + \beta\underline{B}\right) \\ &\quad + \left[\frac{1-\hat{\beta}^{T-1}}{1-\hat{\beta}}\right]\beta\pi\frac{u(Zy)}{1-\beta} + \hat{\beta}^{T-1}\beta\frac{u\left(y - (1-\beta)\underline{B}\right)}{1-\beta}. \end{aligned}$$

To prove part 1 of the Lemma, observe that optimality conditions for B_{t+1} in $t = 0, ..., T-1$ yield

$$\begin{aligned} u'\left(y - B_0 + \hat{\beta}B_1\right) = u'\left(y - B_1 + \hat{\beta}B_2\right) &\quad\Rightarrow\quad g_0 = g_1 \\ \vdots &\qquad\qquad \vdots \\ u'\left(y - B_{T-2} + \hat{\beta}B_{T-1}\right) = u'\left(y - B_{T-1} + \beta\underline{B}\right) &\quad\Rightarrow\quad g_{T-2} = g_{T-1}. \end{aligned}$$

Therefore, $g_{t-1} = g_t$ for every $t = 1, ..., T-1$. In other words, the government smooths spending while it decreases debt to reach at the No-Default region.

To prove part 2, recall that if the government exits the Crisis region in T periods, then it must be that $B_t > \underline{B}$ for every $t \leq T-1$. If this was not the case, then the government would have already exited the Crisis region at an earlier $\tilde{T} < T$, thus contradicting the assumption that running debt in T periods was optimal. Hence, at $t = T-1$, $B_{T-1} > \underline{B}$ implies $g_{T-1} < \bar{g}$. But, since the government smooths spending for T periods, $g_t = g_{T-1}$ implies

$$g_t < \bar{g} \quad \forall t = 0, 1, ..., T-1.$$

Therefore, g_t is smaller than the optimal level of spending that the government attains at the No-Default region, $\bar{g}$. ■

We now turn to the proof of Proposition (2), that follows a similar logic to the result established by Cole and Kehoe (2000).

Proof of Proposition (2). The idea of the proof is to show that for some $\pi > 0$ there is an initial range of debt levels in the Crisis region $\left[\underline{B}, \bar{B}(\pi)\right]$ where the government strictly prefers to run down debt in one period. As B_0 increases, we then show that $T^*(B_0)$ monotonically increases; namely, that the government delays exiting the Crisis region as its outstanding obligations rise. Therefore, for a higher B_0 in $\left[\underline{B}, \bar{B}(\pi)\right]$, this result implies that the government optimally lowers initial debt following policies $T^*(B_0) \in \{1, 2, ..., \infty\}$. Throughout the proof, we assume that $\pi > 0$ and that the government starts in the Crisis region ($B_0 > \underline{B}$). We also assume that the government chooses to run down debt in T periods ($B_T = \underline{B}$).[10]

To prove that $T^*(B_0) = 1$ strictly dominates any other plan $T > T^*(B)$ for some interval in $\left[\underline{B}, \bar{B}(\pi)\right]$, we use the value function when $T = 1$,

$$V^1(B_0) = u\left(y - B_0 + \beta \underline{B}\right) + \beta \frac{u(\bar{g})}{1-\beta}.$$

The value function when $T > 1$ (evaluated at the optimal path $\{B_{t+1}\}$ is

$$V^T(B_0) = \left(1 + \hat{\beta} + ... + \hat{\beta}^{T-1}\right) u\left(g^T(B_0)\right) + \hat{\beta}^{T-1} \beta \frac{u(\bar{g})}{1-\beta} + \left[\text{Value if default occurs}\right],$$

with

$$g^T(B_0) = y - \frac{1-\hat{\beta}}{1-\hat{\beta}^T} B_0 + \frac{1-\hat{\beta}}{1-\hat{\beta}^T} \hat{\beta}^{T-1} \beta \underline{B}.$$

Starting at $t = 0$ and if $T = 1$, the government attains $\bar{g}$ when $t = 1$. For any other policy $T > 1$, however, government spending at $t = 1$ is still at $g^T(B_0)$. Using that $g^T(B_0) < \bar{g}$ (Lemma 12), this means that the continuation value of $V^1(B_0)$ is greater than $V^T(B_0)$ for any $T > 1$. If we then compare government spendings at $t = 0$, we know that g_0 under policy $T = 1$ will be strictly greater than any $g^T(B_0)$ if

$$y - B_0 + \beta \underline{B} > y - \frac{1-\hat{\beta}}{1-\hat{\beta}^T} B_0 + \frac{1-\hat{\beta}}{1-\hat{\beta}^T} \hat{\beta}^{T-1} \beta \underline{B},$$

from where it follows that

$$B_0 < \frac{\underline{B}}{1-\pi}.$$

Therefore, if $B_0 \in \left(\underline{B}, \underline{B}/(1-\pi)\right)$, $V^1(\cdot)$ dominates any other value function $V^T(\cdot)$, for $T > 1$.

It is easy to show that $V^1(B_0) > V^2(B_0) > ... > V^\infty(B_0)$ when $B_0 = \underline{B}/(1-\pi)$. First, note that

$$V^1(B_0) = u(\underbrace{y - B_0 + \beta \underline{B}}_{g^1}) + \beta \frac{u\left(y - (1-\beta)\underline{B}\right)}{1-\beta},$$

$$V^2(B_0) = \left(1 + \hat{\beta}\right) u(\underbrace{y - \frac{B_0}{1+\hat{\beta}} + \frac{1}{1+\hat{\beta}} \hat{\beta} \beta \underline{B}}_{g^2}) + \beta \pi \frac{u(Zy)}{1-\beta} + \hat{\beta} \beta \frac{u\left(y - (1-\beta)\underline{B}\right)}{1-\beta}.$$

[10] Here, we skip some intermediate results from Cole and Kehoe (2000). In their proof, the authors analytically show that (i) $T^*(B_0)$ increases by one period as B_0 increases, and (ii) there cannot be sudden jumps from a finite $T^*(B_0)$ to $T^*(B_0) \to \infty$.

By comparing $g^1(B_0)$ and $g^2(B_0)$ at $B_0 = \underline{B}/(1-\pi)$, we obtain that government spending in the first period are the same,

$$g^1(B_0) = y - \underline{B}\left[\frac{1-\hat{\beta}}{1-\pi}\right] = g^2(B_0).$$

Using Lemma 12, it is easy to observe that $g^1(B_0)$ and $g^2(B_0)$ are strictly below $\bar{g}$ for any $\pi > 0$,

$$g^1(B_0) = g^2(B_0) = y - \underline{B}\left[\frac{1-\hat{\beta}}{1-\pi}\right] < y - (1-\beta)\underline{B} = \bar{g},$$

But, from period $t = 1$ onward, $V^1(B_0)$ outperforms $V^2(B_0)$ since $V^1(B_0)$ reaches $\bar{g}$ one period before $V^2(B_0)$ – Lemma 12, part 2, guarantees that $g^2(B_0) < \bar{g}$, thus making $V^1(B_0)$ to strictly dominate $V^2(B_0)$. By the same token, we can extend this logic to argue that $V^1(B_0)$ dominates $V^T(B_0)$ for any $T > 1$. Hence, $T^*(B_0) = 1$ is preferred to any other policy T when $B_0 = \underline{B}/(1-\pi)$.

To prove that every payoff $V^T(B_0)$ is ranked in decreasing order at $B_0 = \underline{B}/(1-\pi)$, take $V^T(B_0)$ and $V^{T+1}(B_0)$, for $T > 1$, and note that $g^T(B_0)$ equals

$$g^T(B_0) = y - \underline{B}\left[\frac{1-\hat{\beta}}{1-\pi}\right].$$

Since $g^T = g^{T+1}$ in periods $0, 1, ..., T$, value $V^T(B_0)$ reaches $\bar{g}$ before any $V^{T+1}(B_0)$, i.e., $V^T(B_0) > V^{T+1}(B_0)$. Thus, the government strictly prefers to run down debt in one period when $B_0 \in [\underline{B}, \underline{B}/(1-\pi)] \subset [\underline{B}, \bar{B}]$.

We now show that schedules $T^*(B_0) > 1$ can also be optimal for debt levels $B_0 > \underline{B}/(1-\pi)$ in the Crisis region. Specifically, as B_0 increases, we need to prove that (i) every value function $V^T(B_0)$ strictly decreases, (ii) value functions $V^T(B_0)$ with higher T decrease by less, and (iii) for small $\pi > 0$, there exists a full range of possibilities $T^*(B_0)$ such that a higher T increases the government payoff $V^T(B_0)$, i.e. $\partial V^T(B_0)/\partial T > 0$. As a result, (i) - (iii) imply that the optimal $T^*(B_0)$ rises as B_0 increases in $[\underline{B}/(1-\pi), \bar{B}]$. The idea behind this part of the proof is that a higher B_0 increases the government's debt burden when it has to run down debt. Therefore, an increase in B_0 should intuitively make delay (namely, a higher T) preferable. In particular, discounting makes the government prefer to divide the decrease of initial debt B_0 into more periods in order to avoid higher initial sacrifices of g_t.

To show (i), note that

$$\frac{d}{dB_0}V^T(B_0) = -u'\left(g^T\right) < 0 \quad \text{and} \quad \frac{d}{dB_0}V^\infty(B_0) = -u'(g^\infty) < 0$$

implies $dV^T(B_0)/dB_0 < 0$ for any $T = 1, 2, ..., \infty$.

To show (ii), compare first any $dV^T(B_0)/dB_0$ and $dV^\infty(B_0)/dB_0$, and propose that

$$\frac{d}{dB_0}V^T(B_0) < \frac{d}{dB_0}V^\infty(B_0) \Rightarrow u'\left(g^T\right) > u'(\bar{g}).$$

It can be easily shown that $g^T < \bar{g}$ when $B_0 > \underline{B}/(1-\pi)$. With a similar logic, $dV^{T-1}(B_0)/dB_0 < dV^T(B_0)/dB_0$, since $u'\left(g^{T-1}\right) > u'\left(g^T\right)$ when $B_0 > \underline{B}/(1-\pi)$. Therefore,

$$0 > \frac{d}{dB_0}V^\infty(B_0) > ... > \frac{d}{dB_0}V^2(B_0) > \frac{d}{dB_0}V^1(B_0). \tag{29}$$

Intuitively, since increasing B_0 worsens any scenario for the government, delaying more is preferable in order to sacrifice less spending in the initial periods (and gain more because of discounting).

To prove (iii), we follow Cole and Kehoe (2000) and assume that T is continuous. Thus, we can take the derivative of $V^T(B_0)$ with respect to T and obtain,

$$\frac{\partial V^T(B_0)}{\partial T} = \hat{\beta}^{T-1}\ln\hat{\beta}\times\left\{\frac{u_{g^T}\left(g^T\right)}{1-\hat{\beta}^T}\left[\beta\underline{B}-\hat{\beta}B_0\right]+\beta\frac{u(\bar{g})}{1-\beta}-\frac{\beta}{1-\hat{\beta}}\left[(1-\pi)\,u\left(g^T(B_0)\right)+(\pi)\frac{u(Zy)}{1-\beta}\right]\right\}. \tag{30}$$

When $B_0 \in (\underline{B}/(1-\pi), \bar{B}]$, the first term (multiplied by $\ln\hat{\beta}$) is positive. This term shows the government gain from remaining in the Crisis region in every period. The price in the Crisis region $\hat{\beta}$, is always lower than the price of the No-Default region, β. However, for some large levels B_0 in the Crisis region, the government gain $\hat{\beta}B_0$ is greater than its gain $\beta\underline{B}$ when it arrives to the No-Default region. As a result, this provides an incentive to delay exit. In contrast, the last terms (multiplied by $\ln\hat{\beta}$) are the cost of staying in the Crisis region. Intuitively, the opportunity cost of remaining in the Crisis region is $\beta u(\bar{g})/(1-\beta)$, namely, the value of not attaining the safer $\bar{g}$ level in the No-Default region. Finally, the last term shows that the government risks payoff $u(Zy)/(1-\beta)$ with probability π when it follows policy T.

If we can find that $\partial V^T(B_0)/\partial T$ is positive for some small π and $B_0 \in (\underline{B}/(1-\pi), \bar{B}]$, then the benefit of delaying exit from the Crisis region is greater than the cost. When this is the case, there exist other possibilities $T > 1$ that will increase the government's payoff when debt lies in the Crisis region.

Taking the limit on the last expression simplifies to

$$\lim_{\pi\to 0}\frac{\partial V^T(B_0)}{\partial T} = \beta^{T-1}\ln(\beta)\left\{\beta\frac{u_{g^T}\left(g^T\right)}{1-\beta^T}\left(\underline{B}-B_0\right)+\frac{\beta}{1-\beta}\left[u(\bar{g})-u\left(g^T(B_0)\right)\right]\right\}.$$

By Lemma 12, $\bar{g} > g^T(B_0)$. Using property of strictly concave functions,

$$u_{g^T}\left(g^T\right)\left(\bar{g}-g^T(B_0)\right) > u(\bar{g})-u\left(g^T(B_0)\right),$$

and since

$$\bar{g}-g^T(B_0) = \frac{1-\beta}{1-\beta^T}\left[B_0-\underline{B}\right],$$

then

$$u_{g^T}\left(g^T\right)\left(\frac{1-\beta}{1-\beta^T}\right)\left[B_0-\underline{B}\right] > u(\bar{g})-u\left(g^T(B_0)\right).$$

Multiplying the last expression by $\ln(\beta)$ changes its sign. Rearranging terms in $\partial V^T(B_0)/\partial T$ and using the last inequality imples

$$\begin{aligned}\lim_{\pi\to 0}\frac{\partial V^T(B_0)}{\partial T} &= \beta^{T-1}\ln(\beta)\left\{\beta\frac{u_{g^T}\left(g^T\right)}{1-\beta^T}\left(\underline{B}-B_0\right)+\frac{\beta}{1-\beta}\left[u(\bar{g})-u\left(g^T(B_0)\right)\right]\right\}\\ &> \beta^{T-1}\left\{\ln(\beta)\frac{\beta}{1-\beta^T}u_{g^T}\left(g^T\right)\left(\underline{B}-B_0\right)+\frac{\beta}{1-\beta}\ln(\beta)\,u_{g^T}\left(g^T\right)\left(\frac{1-\beta}{1-\beta^T}\right)\left[B_0-\underline{B}\right]\right\}\\ &= 0.\end{aligned}$$

Notice then that, for some B_0 and π very small, $\partial V^T(B_0)/\partial T > 0$. Finally, the sign of equations (29) and (30) implicitly define a positive relation of T in terms of B_0, thus proving that $T^*(B_0)$

can take values $\{1, 2, ..., \infty\}$ for small values of π, as B_0 increases in $\left[\underline{B}, \bar{B}(\pi)\right]$. ■

7.3.10 Relation between $g^T(B_0)$ and $g^{T+1}(B_0)$

Claim 13 *If $B_0 > \underline{B}/(1-\pi)$, then $g^T(B_0) < g^{T+1}(B_0)$.*

Proof. Be $g^T(B_0) < g^{T+1}(B_0)$. Then

$$y - \left[\frac{1-\hat{\beta}}{1-\hat{\beta}^T}\right] B_0 + \left[\frac{1-\hat{\beta}}{1-\hat{\beta}^T}\right] \hat{\beta}^{T-1} \beta \underline{B} < y - \left[\frac{1-\hat{\beta}}{1-\hat{\beta}^{T+1}}\right] B_0 + \left[\frac{1-\hat{\beta}}{1-\hat{\beta}^{T+1}}\right] \hat{\beta}^T \beta \underline{B}.$$

After rearranging terms,

$$0 < \hat{\beta}^{T-1}\left(B_0\hat{\beta} - \beta\underline{B}\right),$$

and this inequality holds only if the term in parenthesis is positive, what, in turn, implies that $B_0 > \underline{B}/(1-\pi)$. ■

7.3.11 Comparative statics (region boundaries with respect to Z)

Proof of Proposition 3. From equation 10 we can write

$$G(y, Z, \beta, \underline{B}) = \frac{u(Zy)}{1-\beta} - u(y-\underline{B}) - \beta\frac{u(y)}{1-\beta} = 0,$$

and after applying the implicit function theorem, we get

$$\frac{\partial \underline{B}}{\partial Z} = -\frac{\frac{u'(Zy)}{1-\beta} y}{u'(y-\underline{B})} < 0.$$

Assuming that V^∞ dominates any other V^T when π is positive, we can write equation (15) as

$$F\left(y, Z, \beta, \pi\bar{B}\right) = \frac{u\left(y - \left(1-\hat{\beta}\right)\bar{B}\right)}{1-\hat{\beta}} + \frac{\beta\pi u(Zy)}{\left(1-\hat{\beta}\right)(1-\beta)} - u\left(Zy + \hat{\beta}\bar{B}\right) - \beta\frac{u(Zy)}{1-\beta} = 0.$$

Using the implicit function theorem we get

$$\frac{\partial \bar{B}}{\partial Z} = -\frac{-\left(\beta\frac{u'(Zy)y}{1-\beta}\left[1-\frac{\pi}{1-\hat{\beta}}\right] + u'\left(Zy + \hat{\beta}\bar{B}\right)y\right)}{-\left(u'\left(y-\left(1-\hat{\beta}\right)\bar{B}\right) + u'\left(Zy+\hat{\beta}\bar{B}\right)\hat{\beta}\right)},$$

and since the term in square brackets in the numerator is positive for any $\pi \in (0,1)$, then $\partial\bar{B}/\partial Z < 0$.

If the government lowers debt in $T = 1$ period,

$$V^1(B_0) = u(y - B_0 + \beta\underline{B}) + \beta\frac{u(y-(1-\beta)\underline{B})}{1-\beta},$$

and

$$\frac{\partial V^1}{\partial Z} = \beta\frac{\partial \underline{B}}{\partial Z}\left[u'(g_0) - u'(\bar{g})\right].$$

Given that $\partial \underline{B}/\partial Z < 0$, and since $g_0 < \bar{g}$ (Lemma 12) implies that $u'(g_0) > u'(\bar{g})$, then $\partial V^1/\partial Z < 0$.

Finally, it follows immediately from V^∞ that

$$\frac{\partial V^\infty (B_0)}{\partial Z} = \frac{\beta \pi u (Zy)}{\left(1-\hat{\beta}\right)(1-\beta)} y > 0.$$

■

Appendix B - Long-term debt

7.4 Determination of the equilibrium price

If the government is in the No-Default zone, then it never defaults ($z' = 1$) and price is stationary ($q = q'$). Hence

$$q\left(s, B'\right) = \beta E\left[z'\left(1 + q'\delta\right)\right] \Rightarrow q\left(s, B'\right) = \frac{\beta}{1 - \beta\delta}$$

If price were stationary in the Crisis region, then government defaults with probability π, and otherwise it repays with probability $1 - \pi$. Therefore, the equilibrium price is now $q = \beta\left[(1-\pi)(1+q\delta) + \pi(0)\right]$ and yields

$$q = \frac{\beta(1-\pi)}{1 - \beta(1-\pi)\delta} = \frac{\hat{\beta}}{1 - \hat{\beta}\delta}$$

where we defined $\hat{\beta} = \beta(1-\pi)$.

Finally, if debt is lowered in T periods, the equilibrium price can be built with the following procedure. We start with $T = 1$ and use a backward induction logic. Conditional on no previous default, the government will reach the No-Default region in $t = 1$, and the stationary price is $q_1^1 = \beta/(1-\beta\delta)$. At the initial period $t = 0$, the q_0^1 comes defined as

$$q_0^1 = \beta E\left[z'\left(1 + q_1^1\delta\right)\right].$$

and, after plugging the previous result, we obtain that $q_0^1 = q_1^1 = \beta/(1-\beta\delta)$. Assume that $T = 2$; then prices in each period are

$$q_0^2 = \beta E\left[z'\left(1 + q_1^2\delta\right)\right] \Rightarrow q_0^2 = \beta\left[(1-\pi)\left(1 + q_1^2\delta\right)\right]$$

$$q_1^2 = \beta E\left[z'\left(1 + q_2^2\delta\right)\right] \Rightarrow \beta\left(1 + q_2^2\delta\right)$$

$$q_2^2 = \frac{\beta}{1 - \beta\delta}.$$

Then, $q_2^2 = q_1^2 = \beta/(1-\beta\delta)$, and

$$q_0^2 = \hat{\beta} + \hat{\beta}\delta\frac{\beta}{1 - \beta\delta}.$$

Since $q_2^2 = q_1^2$ also occurs when $T = 1$ (namely, that $q_1^1 = q_0^1$), we can already identify that $q_{t-1}^T = q_t^T$; i.e., that $q_t^T = \beta/(1-\beta\delta)$ when $t \geq T - 1$.

We do the same analysis when $T = 3$, with the only difference that we now disregard q_3^3 since it will be equal to q_2^3. Therefore,

$$q_0^3 = \beta E\left[z'\left(1 + q_1^3\delta\right)\right] = \beta\left[(1-\pi)\left(1 + q_1^3\delta\right)\right]$$

$$q_1^3 = \beta E\left[z'\left(1 + q_2^3\delta\right)\right] = \beta\left[(1-\pi)\left(1 + q_2^3\delta\right)\right]$$

$$q_2^3 = \beta E\left[z'\left(1 + q_3^3\delta\right)\right] = \beta\left(1 + q_3^3\delta\right).$$

After some algebra, this results in $q_2^3 = \beta/(1-\beta\delta)$, $q_1^3 = \hat{\beta} + \hat{\beta}\delta\beta/(1-\beta\delta)$, and $q_0^3 = \hat{\beta}(1 + \hat{\beta}\delta) +$

$(\hat{\beta}\delta)^2\beta/(1-\beta\delta)$. Finally, this allows us to identify that

$$q_0^T = \hat{\beta}\left(\sum_{k=0}^{T-2}\left(\hat{\beta}\delta\right)^k\right) + \left(\hat{\beta}\delta\right)^{T-1}\left(\frac{\beta}{1-\beta\delta}\right),$$

and, therefore, the equilibrium price is

$$q_t^T = \begin{cases} \left(1-\left(\hat{\beta}\delta\right)^{T-1-t}\right)\left(\frac{\hat{\beta}}{1-\hat{\beta}\delta}\right) + \left(\hat{\beta}\delta\right)^{T-1-t}\left(\frac{\beta}{1-\beta\delta}\right) & t < T-1 \\ \frac{\beta}{1-\beta\delta} & t \geq T-1 \end{cases}$$

7.5 Value of repayment and determination of the lower bound of the Crisis region under long-term debt

7.5.1 Value of repayment when the credit market freezes for a finite number of periods

We need two results to derive V^R under long-term debt.[11] Claim 14 states that the government chooses a constant level of spending when there is no probability of default (i.e., $\pi = 0$). Claim 15 states that if government spending is constant, the path $\{B_{t+1}\}$ is constant.

Claim 14 *If $\pi = 0$, then government spending is constant.*

Proof. When $\pi = 0$, the equilibrium price of a riskless bond is fixed and equal to $q = \beta/(1-\beta\delta)$. Hence, the per-period budget constraint is

$$g_t + B_t = y + \frac{\beta}{1-\beta\delta}\left(B_{t+1} - \delta B_t\right) \Rightarrow g_t = y + \frac{\beta}{1-\beta\delta}B_{t+1} - \frac{1}{1-\beta\delta}B_t,$$

and the government chooses the sequence $\{B_{t+1}\}_{t=0}^{\infty}$ that solves

$$\max \sum_{t=0}^{\infty} \beta^t u\left(g_t\right)$$

$$\begin{aligned} s.t. \quad & g_t = y + \tfrac{\beta}{1-\beta\delta}B_{t+1} - \tfrac{1}{1-\beta\delta}B_t \quad \forall t \geq 0 \\ & B_t = B \text{ given, and } B \leq \underline{B} \\ & B_{t+1} \leq \underline{B} \quad \forall t \geq 0. \end{aligned}$$

First-order and transversality conditions yield

$$\beta^t u'\left(g_t\right)\left[\frac{\beta}{1-\beta\delta}\right] + \beta^{t+1}u'\left(g_{t+1}\right)\left[-\frac{1}{1-\beta\delta}\right] = 0$$

$$\lim_{s\to\infty} \beta^s \lambda_s B_{s+1} = 0.$$

From the first expression we obtain that $u'\left(g_t\right) = u'\left(g_{t+1}\right)$, implying that government spending is constant. ■

Claim 15 *If g_t is constant and initial debt B_0 is riskless, then $B_t = B_0$ for every t.*

[11] These results particularize intermediate steps in Claim 7 for long-term debt.

Proof. Take

$$\begin{aligned} g_t &= y + \frac{\beta}{1-\beta\delta} B_{t+1} - \frac{1}{1-\beta\delta} B_t \\ g_{t+1} &= y + \frac{\beta}{1-\beta\delta} B_{t+2} - \frac{1}{1-\beta\delta} B_{t+1}. \end{aligned}$$

Using that $g_t = g_{t+1}$ from Claim 14 leads to

$$B_{t+2} - B_{t+1} = \frac{1}{\beta}\left(B_{t+1} - B_t\right).$$

By a logic similar to Claim 7, we can argue that $B_{t+1} = B_t$ for every t. ■

We can now build the value of repayment when the credit market freezes for T periods.

Claim 16 *The value of repayment when the market freezes for T periods is given by*

$$V^R(B_0, \delta) = \sum_{t=0}^{\infty} \beta^t u(g_t) \quad s.t.\ g_t = \begin{cases} y - \delta^T B_0 & t = T \\ y - \frac{1-\beta}{1-\beta\delta}\delta^T B_0 & t > T \end{cases}$$

Proof. In this problem, we are looking for the highest B_0 such that the government does not default in the No-Default region (i.e., when $\pi = 0$). Hence, we divide the problem in two parts. The first part is the government problem when the market freezes lending for T periods, namely, for $t = 0, 1, ..., T-1$. The second part is when the market resumes lending for $t = T, T+1, ...$. We use a backward induction logic, starting from the second problem and then moving to the first problem.

Starting at $t \geq T$, by Claim 14 the government spending is constant in every period. Also, Claim 15 implies that debt is constant and equal to the debt level at the initial period. Since the initial period in this problem is the date when the credit market resumed lending ($t = T$), then debt will be constant at $B_t = B_T$. Therefore, government spending will be equal to

$$g = y - \frac{1-\beta}{1-\beta\delta} B_T,$$

and the value after the market resumes lending is

$$\frac{u\left(y - \frac{1-\beta}{1-\beta\delta} B_T\right)}{1-\beta}.$$

In the first part of the problem, there is no lending for $t = 0, 1, ..., T-1$ periods. Since the government collects nothing from its issuances, $i_t = 0$ implies $B_{t+1} = \delta B_t$ $t = 0, ..., T-1$. The iteration of the last expression yields $B_t = \delta^t B_0$. Finally, the government's payoff in the first problem is

$$\sum_{t=0}^{T-1} \beta^t u\left(y - \delta^t B_0\right),$$

and the government value of repayment when there is a T-period market freeze is

$$V^R(B_0, \delta) = \sum_{t=0}^{T-1} \beta^t u\left(y - \delta^t B_0\right) + \beta^T \frac{u\left(y - \frac{1-\beta}{1-\beta\delta}\delta^T B_0\right)}{1-\beta}.$$

It is worth noting that after a market freeze, lending resumes. But the government has already achieved a lower level of debt, and by Claim 15 it will keep that level constant. Since the level of debt was the highest at the initial period, then if the government has not defaulted before, it will surely not default after lending resumes. Therefore, we can use this value of repayment to compare it later against the value of default at equality, in order to characterize the highest B. ■

7.5.2 Determination of the lower bound of the crisis region

Having identified the No-Lending Condition under long-term debt, we now prove that there exists a unique solution such that the equation holds with equality under a one-period market freeze.

Lemma 17 *When $T = 1$, there exists a unique B that is solution to*

$$\frac{u(Zy)}{1-\beta} = u(y-B) + \beta \frac{u\left(y - \frac{1-\beta}{1-\beta\delta}\delta B\right)}{1-\beta}$$

Proof. We start building

$$F(B; Z, y, \beta, \delta) = \frac{u(Zy)}{1-\beta} - u(y-B) - \beta \frac{u\left(y - \frac{1-\beta}{1-\beta\delta}\delta B\right)}{1-\beta} = 0$$

Provided that (i) $F(0; \cdot) = \frac{1}{1-\beta}\left[u(Zy) - u(y)\right] < 0$, (ii) $F(B \to y; \cdot) = u(Zy)/(1-\beta) - u(0) - \beta u(y(1-\delta)/(1-\beta\delta))/(1-\beta) \to \infty$, and (iii) $F'(B) = u'(y-B) + \delta\beta u'(y - \delta B(1-\beta)/(1-\beta\delta))/(1-\beta\delta) > 0$, then we are guaranteed that a maximum $\underline{B}(Z, y, \beta, \delta) > 0$ exists and is unique. ■

7.6 Determination of the upper bound of the Crisis region

7.6.1 Optimal policy in the No-Default region under long-term debt

Claim 18 (Long-term debt) *If $\pi = 0$ and given an initial debt $B_{t'} = B \leq \underline{B}$, then government spending is constant and equal to*

$$g = y - B\left(\frac{1-\beta}{1-\beta\delta}\right),$$

and government debt B_{s+1} equals the initial debt $B_{t'}$ for $s \geq t'$.

Proof. This problem is identical to the government problem in Claim 14 (which we can particularize for any $t = t'$ instead of $t = 0$ as in Claim 7). Moreover, we can use the result in Claim 15 to establish that debt is constant and equal to the initial debt. After setting a constant g and B in the budget constraint, the optimal government spending is

$$g = y - \left(\frac{1-\beta}{1-\beta\delta}\right) B_0.$$

■

7.6.2 General problem when debt is lowered in T periods (long-term debt)

The government problem can be written as

$$\begin{aligned} V^T\left(B_0,\delta\right) &= \max_{\{B_{t+1}\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \hat{\beta}^t u\left(y - B_t\left(1+\delta q_t^T\right) + q_t^T B_{t+1}\right) \\ &\quad + \hat{\beta}^{T-1} u\left(y - B_{T-1}\left(1+\delta q_{T-1}^T\right) + q_{T-1}^T \underline{B}\right) \\ &\quad + \left[\frac{1-\hat{\beta}^{T-1}}{1-\hat{\beta}}\right] \beta\pi \frac{u\left(Zy\right)}{1-\beta} + \hat{\beta}^{T-1}\beta \frac{u\left(y - \left(\frac{1-\beta}{1-\beta\delta}\right)\underline{B}\right)}{1-\beta} \end{aligned}$$

subject to

$$\begin{aligned} q_t^T &= \beta E\left[z_{t+1}\left(1+q_{t+1}^T\delta\right)\right], \quad z_{-1}=1 \\ & B_0, \quad \underline{B} \quad \textit{given.} \end{aligned}$$

The first-order condition yields

$$u'\left(g_t\right) = u'\left(g_{t+1}\right) \frac{\hat{\beta}\left(1+\delta q_{t+1}^T\right)}{q_t^T}$$

Given that B_0 is in the Crisis region, and since every debt level $\{B_{t+1}\}$ lies on it while the government plans to exit the region in T periods,[12] the equilibrium price relationship $q_t^T = \hat{\beta}\left(1+\delta q_{t+1}^T\right)$ holds, and as a result we have

$$u'\left(g_t\right) = u'\left(g_{t+1}\right) \Rightarrow g_t = g_{t+1}$$

7.6.3 Expression for $g^T\left(B_0,\delta\right)$

We assume that the government is in the crisis region ($B > \underline{B}$), and use every budget constraint when $z=1$. Writing $g^T\left(B_0\,,\delta\right) \equiv g^T$ yields

$$g^T + B_t = y + q_t^T\left(B_{t+1} - B_t\delta\right).$$

We initially solve for B_0 for different T schemes in order to identify a pattern. After that, we calculate a general formula for g^T.

When $T=2$, the budget constraints during the transition are

$$g^2 + B_0 = y + q_0^2\left(B_1 - B_0\delta\right) \Rightarrow B_0 = \frac{y-g^2}{1+q_0^2\delta} + \frac{q_0^2}{1+q_0^2\delta}B_1$$

$$g^2 + B_1 = y + q_1^2\left(\underline{B} - B_1\delta\right) \Rightarrow B_1 = \frac{y-g^2}{1+q_1^2\delta} + \frac{q_1^2}{1+q_1^2\delta}\underline{B}.$$

And after plugging B_1 into B_0,

$$B_0 = \frac{y-g^2}{1+q_0^2\delta}\left[1 + \frac{q_0^2}{1+q_1^2\delta}\right] + \frac{q_0^2}{1+q_0^2\delta}\frac{q_1^2}{1+q_1^2\delta}\underline{B}.$$

[12] If not, exit from the Crisis region would have occured for some B_{t+1} in the optimal sequence, contradicting the fact that the government exits in T.

Doing the similar proces for $T = 3$ and $T = 4$, we obtain that

$$B_0 = \frac{y - g^3}{1 + q_0^3 \delta} \left[1 + \frac{q_0^3}{1 + q_1^3 \delta} + \frac{q_0^3}{1 + q_1^3 \delta} \frac{q_1^3}{1 + q_2^3 \delta} \right] + \frac{q_0^3}{1 + q_0^3 \delta} \frac{q_1^3}{1 + q_1^3 \delta} \frac{q_2^3}{1 + q_2^3 \delta} \underline{B}$$

$$\begin{aligned} B_0 &= \frac{y - g^4}{1 + q_0^4 \delta} \left[1 + \frac{q_0^4}{1 + q_1^4 \delta} + \frac{q_0^4}{1 + q_1^4 \delta} \frac{q_1^4}{1 + q_2^4 \delta} + \frac{q_0^4}{1 + q_1^4 \delta} \frac{q_1^4}{1 + q_2^4 \delta} \frac{q_2^4}{1 + q_3^4 \delta} \right] \\ &\quad + \frac{q_0^4}{1 + q_0^4 \delta} \frac{q_1^4}{1 + q_1^4 \delta} \frac{q_2^4}{1 + q_2^4 \delta} \frac{q_3^4}{1 + q_3^4 \delta} \underline{B}. \end{aligned}$$

Hence, for any T we have that

$$B_0 = \frac{y - g^T}{1 + q_0^T \delta} \left[1 + \sum_{k=0}^{T-2} \frac{\prod_{j=0}^{T-2-k} q_j^T}{\prod_{j=1}^{T-1-k} \left(1 + q_j^T \delta \right)} \right] + \frac{\prod_{j=0}^{T-1} q_j^T}{\prod_{j=0}^{T-1} \left(1 + q_j^T \delta \right)} \underline{B},$$

and after solving for g^T,

$$g^T (B_0, \delta) = y - \left(1 + q_0^T \delta \right) \left[B_0 - \frac{\prod_{j=0}^{T-1} q_j^T}{\prod_{j=0}^{T-1} \left(1 + q_j^T \delta \right)} \underline{B} \right] \left[1 + \sum_{k=0}^{T-2} \frac{\prod_{j=0}^{T-2-k} q_j^T}{\prod_{j=1}^{T-1-k} \left(1 + q_j^T \delta \right)} \right]^{-1}.$$

7.6.4 The upper bound of the Crisis region when $\pi = 0$

Lemma 19 *Suppose that $\pi \to 0$, and consider $V^{\infty} (B_0, \delta) - V^T (B_0, \delta)$, where $\delta > 0$. Then,*

$$V^{\infty} (B_0, \delta) - V^T (B_0, \delta) > 0$$

for every $T < \infty$.

Proof. We use the formula for the prices under long-term debt, q_t^T. As π tends to 0, each $\hat{\beta}$ term is equal to β, and therefore, $q_t^T = \beta / (1 - \beta \delta)$. Labelling $q_t^T \equiv q$ and plugging it into our expression for $g^T (B, \delta)$, we get that

$$g^T (B_0, \delta) = y - (1 + q\delta) \left[B_0 - \left(\frac{q}{1 + q\delta} \right)^T \underline{B} \right] \left[\frac{1 - \frac{q}{1+q\delta}}{1 - \left(\frac{q}{1+q\delta} \right)^T} \right].$$

Since $1 + q\delta = 1 / (1 - \beta\delta)$ and $q = \beta (1 + q\delta)$, then

$$g^T (B_0, \delta) = y - \left[\frac{1 - \beta}{1 - \beta^T} \right] \left[B_0 - \beta^T \underline{B} \right] \left(\frac{1}{1 - \beta\delta} \right),$$

(and it is straightforward to observe that $\delta = 0$ leads to the same expression as our former $g^T (B, 0) \equiv g^T (B)$).

Equation $V^{\infty}(B_0,\delta) - V^1(B_0,\delta)$ is equal to

$$\frac{1}{1-\beta}\left[u\left(y-\frac{1-\beta}{1-\beta\delta}B_0\right)-(1-\beta)\,u\left(y-\frac{B_0}{1-\beta\delta}+\frac{\beta\underline{B}}{1-\beta\delta}\right)+\beta u\left(y-\frac{1-\beta}{1-\beta\delta}\underline{B}\right)\right].$$

Since the argument of the first term is equal to the (convex combination) of the second and third terms' arguments, we use strict concavity in $u(\cdot)$ with Jensen's inequality to obtain that $V^{\infty}(B_0,\delta) - V^1(B_0,\delta) > 0$. Equation $V^{\infty}(B_0,\delta) - V^T(B_0,\delta)$ when $T \geq 2$ yields

$$\begin{aligned} V^{\infty}(B_0,\delta) - V^T(B_0,\delta) &= \frac{1}{1-\beta}[u\left(y-\frac{1-\beta}{1-\beta\delta}B_0\right) - \left[1-\beta^T\right]u\left(g^T(B_0,\delta)\right) \\ &\quad -\beta^T u\left(y-\left(\frac{1-\beta}{1-\beta\delta}\right)\underline{B}\right)], \end{aligned}$$

and provided that $\left(1-\beta^T\right)g^T(B_0,\delta)+\beta^T\left[y-\left((1-\beta)/(1-\beta\delta)\right)\underline{B}\right]$ equals $y-[(1-\beta)/(1-\beta\delta)]\,B_0$, then strict concavity and Jensen's inequality proves that the difference is strictly positive for any debt level. ■

Lemma 20 *There exists a unique $\bar{B}$ that solves*

$$\frac{u\left(y-\frac{1-\beta}{1-\beta\delta}\bar{B}\right)}{1-\beta} = u\left(Zy+\frac{\beta}{1-\beta\delta}\bar{B}\right)+\beta\frac{u(Zy)}{1-\beta}$$

Proof. We define the following function

$$H(B;\beta,y,Z,\delta) = \frac{u\left(y-\frac{1-\beta}{1-\beta\delta}\bar{B}\right)}{1-\beta} - u\left(Zy+\frac{\beta}{1-\beta\delta}\bar{B}\right)-\beta\frac{u(Zy)}{1-\beta} = 0$$

Given that (i) $H(0;\cdot) = \frac{1}{1-\beta}\left[u(y)-u(Zy)\right] > 0$, (ii) $u(0)/(1-\beta) - u(Zy + y\beta/(1-\beta)) - \beta u(Zy)/(1-\beta) \to -\infty$, and (iii) $\partial H(B;\cdot)/\partial B = -u'\left(y-\bar{B}(1-\beta)/(1-\beta\delta)\right)/(1-\beta\delta) - u'\left(Zy+\bar{B}\beta/(1-\beta\delta)\right)(\beta/(1-\beta\delta)) < 0$, then there exists a unique $\bar{B}$ such that the PC is satisfied with equality at $\pi \to 0$. ■

7.6.5 Figure 6

Using the PC and the NLC, define the $U_{PC}^{\pi=0}(B;\delta) \equiv U_{PC}(B;\delta)$ and $U_{NLC}^{\pi=0}(B;\delta) \equiv U_{NLC}(B;\delta)$ curves as

$$U_{PC}(B;\delta) = \frac{u\left(y-\frac{1-\beta}{1-\beta\delta}\bar{B}\right)}{1-\beta} - u\left(Zy+\frac{\beta}{1-\beta\delta}\bar{B}\right)-\beta\frac{u(Zy)}{1-\beta}$$

$$U_{NLC}(B;\delta) = u(y-B)+\beta\frac{u\left(y-\frac{1-\beta}{1-\beta\delta}\delta B\right)}{1-\beta}-\frac{u(Zy)}{1-\beta}.$$

Using the same properties as in Lemmas 17 and 20, then it is straightforward that (i) $U_{PC}(B;\delta) = U_{NLC}(B) > 0$ at $B=0$, (ii) $U_{PC}(B;\delta) \to -\infty$ as $B \to y(1-\beta\delta)/(1-\beta)$ and $U_{NLC}(B;\delta) \to -\infty$ as $B \to y$, and (iii) both $U'_{PC}(B;\delta) < 0$ and $U'_{NLC}(B) < 0$. This describes the entire behavior of the curves in Figure 6. The behavior of $U_{PC}^{\pi\geq 0}(B;\delta)$ is the only relevant for this graph – $U_{NLC}(B;\delta)$

is independent of π. However, we can again invoke Lemma 21 to argue that (i) $U_{PC}^{\pi\geq 0}(0;\delta)>0$, (ii) $U_{PC}^{\pi\geq 0}(B;\delta)<0$ as $B\rightarrow y\left(1-\hat{\beta}\delta\right)/(1-\beta\delta)$, and (iii) $U_{PC}^{\prime\pi\geq 0}(B;\delta)<0$.

7.6.6 The upper bound of the Crisis region when $\pi=0$

Lemma 21 *There exists a unique $\bar{B}(\delta)$ that solves*

$$\frac{u\left(y-B\left[\frac{1-\hat{\beta}}{1-\hat{\beta}\delta}\right]\right)}{1-\hat{\beta}}+\frac{\beta\pi u(Zy)}{\left(1-\hat{\beta}\right)(1-\beta)}=u\left(Zy+\frac{\hat{\beta}}{1-\hat{\beta}\delta}B\right)+\beta\frac{u(Zy)}{1-\beta}$$

Proof. With a similar logic as before, we define

$$\tilde{H}(B;\beta,y,Z,\pi,\delta)=\frac{u\left(y-B\left[\frac{1-\hat{\beta}}{1-\hat{\beta}\delta}\right]\right)}{1-\hat{\beta}}+\frac{\beta\pi u(Zy)}{\left(1-\hat{\beta}\right)(1-\beta)}-u\left(Zy+\frac{\hat{\beta}}{1-\hat{\beta}\delta}B\right)-\beta\frac{u(Zy)}{1-\beta}.$$

Given that (i) $\tilde{H}(0;\cdot)=[u(y)-u(Zy)]/(1-\hat{\beta})>0$, (ii) $\tilde{H}(B\rightarrow y(1-\hat{\beta}\delta)/(1-\hat{\beta});\cdot)=u(0)/(1-\hat{\beta})+\beta\pi u(Zy)/[(1-\hat{\beta})(1-\beta)]-u(Zy+y\hat{\beta}/(1-\hat{\beta}))-\beta u(Zy)(1-\beta)\rightarrow-\infty$, and (iii) $d\tilde{H}(B;\cdot)/dB=-u(y-B(1-\hat{\beta})/(1-\hat{\beta}\delta))/(1-\hat{\beta}\delta)-u(Zy+B\hat{\beta}/(1-\hat{\beta}\delta))\hat{\beta}/(1-\hat{\beta}\delta)<0$, then there exists a positive and unique $\bar{B}(\beta,y,Z,\pi,\delta)$ that satisfies the PC with equality. ■

7.7 Welfare analysis results

7.7.1 Proposition 3

Proof of Proposition 3. We rewrite equation (24) in terms of total outstanding debt and define

$$H=\frac{u(Zy)}{1-\beta}-u\left(y-(1-\delta)\underline{B}\right)-\beta\frac{u\left(y-\frac{1-\beta}{1-\beta\delta}\delta(1-\delta)\underline{B}\right)}{1-\beta},$$

to apply the implicit function theorem. Therefore,

$$\frac{dH}{d\underline{B}}=(1-\delta)u'\left(y-(1-\delta)\underline{B}\right)+\beta u'\left(y-\frac{1-\beta}{1-\beta\delta}\delta(1-\delta)\underline{B}\right)\left(\frac{\delta(1-\delta)}{1-\beta\delta}\right),$$

and

$$\frac{dH}{d\delta}=-B\left[u'\left(y-(1-\delta)\underline{B}\right)-\beta u'\left(y-\frac{1-\beta}{1-\beta\delta}\delta(1-\delta)\underline{B}\right)\left[\frac{1-2\delta+\beta\delta^2}{(1-\beta\delta)^2}\right]\right].$$

The first derivative is positive. Also, expression $y-(1-\delta)\underline{B}<y-(1-\beta)\delta(1-\delta)\underline{B}/(1-\beta\delta)$ for every $\delta\in(0,1)$, and then $u'(y-(1-\delta)\underline{B})>u'(y-(1-\beta)\delta(1-\delta)\underline{B}/(1-\beta\delta))$. Since the inner-most term in brackets is strictly less than 1, the outer-most expression in brackets is positive. Therefore, $dH/d\delta<0$. As a result, $d\underline{B}/d\delta=-(dH/d\delta)/(dH/d\underline{B})>0$.

Using equation (27), we also rewrite it in terms of total outstanding debt to get

$$G=\frac{u\left(y-\frac{1-\hat{\beta}}{1-\hat{\beta}\delta}(1-\delta)\bar{B}\right)}{1-\hat{\beta}}+\frac{\beta\pi}{1-\hat{\beta}}\frac{u(Zy)}{1-\beta}-u\left(Zy+\frac{\hat{\beta}}{1-\hat{\beta}\delta}(1-\delta)\bar{B}\right)-\beta\frac{u(Zy)}{1-\beta}=0.$$

Applying the implicit function theorem,

$$\frac{dG}{dB} = -u'\left(y - \frac{1-\hat{\beta}}{1-\hat{\beta}\delta}(1-\delta)\bar{B}\right)\left(\frac{1-\delta}{1-\hat{\beta}\delta}\right) - u'\left(Zy + \frac{\hat{\beta}}{1-\hat{\beta}\delta}(1-\delta)\bar{B}\right)\left(\frac{\hat{\beta}(1-\delta)}{1-\hat{\beta}\delta}\right) < 0$$

and

$$\frac{dG}{d\delta} = \frac{u'\left(y - \frac{1-\hat{\beta}}{1-\hat{\beta}\delta}(1-\delta)\bar{B}\right)}{1-\hat{\beta}}\bar{B}\left(\frac{1-\hat{\beta}}{1-\hat{\beta}\delta}\right)^2 + u'\left(Zy + \frac{\hat{\beta}}{1-\hat{\beta}\delta}(1-\delta)\bar{B}\right)\left[\frac{\hat{\beta}\bar{B}\left(1-\hat{\beta}\right)}{\left(1-\hat{\beta}\delta\right)^2}\right] > 0$$

result in $d\bar{B}/d\delta = -\left(dG/d\delta\right)/\left(dG/d\bar{B}\right) > 0$. ■

7.7.2 Figure 9

Under short-term debt, Figure 10 graphically shows the envelope function $\hat{V}$ (solid line) that results from the value function $V^5(B_0)$ (dotted line). Specifically, the government chooses $T(B_0) = 5$ approximately when $\hat{B} \in (10.23, 10.38)$. In this interval of total debt levels $\hat{B}$, the envelope function $\hat{V}(B_0)$ matches $V^5(B_0)$; after that interval the government switches to policy $T(B_0) = 6$, making $\hat{V}(B_0)$ lie strictly above $V^5(B_0)$.

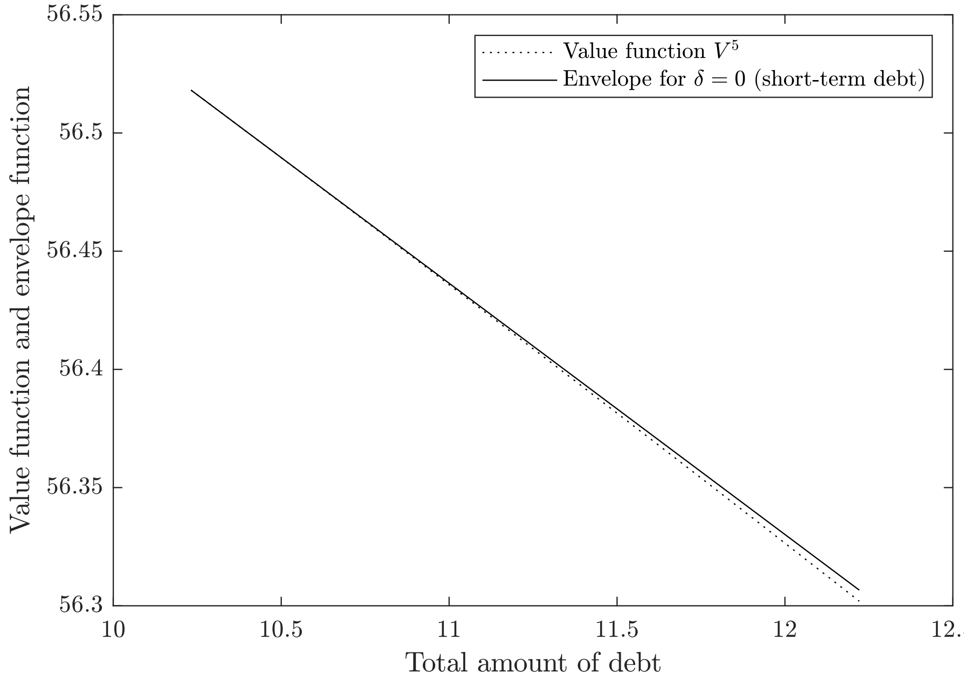

Fig. 10. Value function V^5 and envelope curve for short-term debt as total amount of debt varies. ($\pi = 0.0001$, $\beta = 0.96$, $\delta = 0$, $Z = 0.9$, $y = 10$ and $u(\cdot) = \ln(\cdot)$.)

The same mechanics applies for the envelope function under long-term debt. The next figure illustrates the maximum payoff that the government attains when it follows the optimal strategies prescribed under equilibrium. Specifically, we concentrate in the value that the government obtains in the No-Default and the Crisis region. Notice first that we can disregard the value that the government obtains in the Default region because when the government starts with initial debt $B > \bar{B}$, the definition of a sunspot equilibrium prescribes that creditors will not lend money. As a result, the price q drops to 0, and the government defaults and gets $V^D = u(Zy)/(1-\beta)$ for every debt level $B \geq \bar{B}$, both under short-term and long-term debt. If we can show that in the Crisis region the envelope under long-term debt is above the envelope under short-term debt, then we can rule out any possibility that the economy attains a higher utility in the Default region when $\delta = 0$.

With this logic, we can numerically prove that welfare under long-term debt is greater compared to short-term debt by showing that (i) in the No-Default region, the government attains a higher utility under long-term debt when it smooths spending, and (ii), in the Crisis region, the envelope of the family of value functions under long-term debt is always above the envelope under short-term debt.

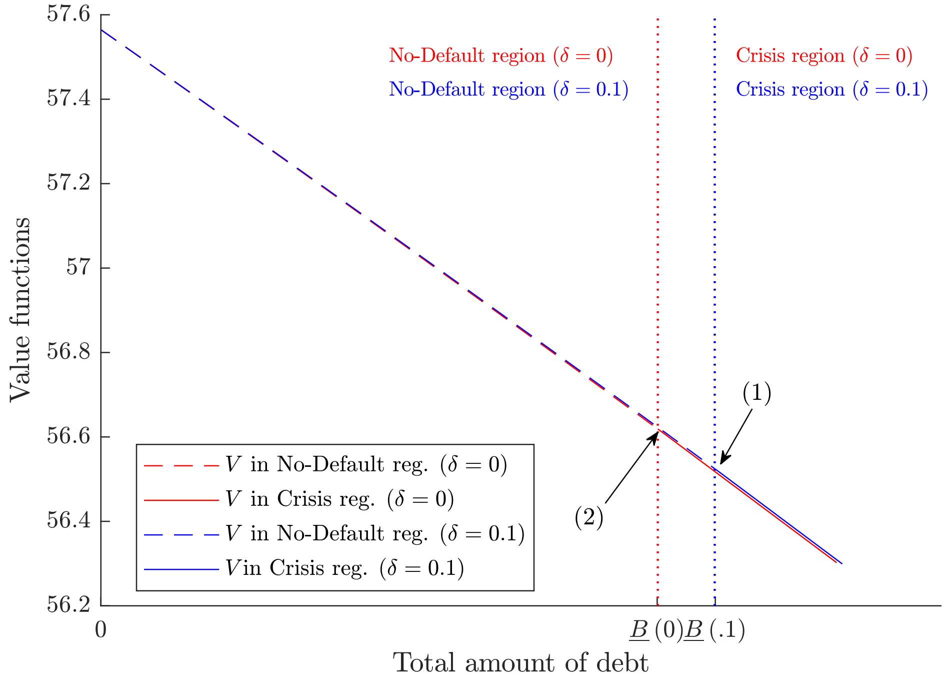

Fig. 11. Debt regions and maximum V as total amount of debt varies, short and long-term debt. ($\pi = 0.0001$, $\beta = 0.96$, $\delta = \{0, 0.1\}$, $Z = 0.9$, $y = 10$ and $u(\cdot) = \ln(\cdot)$)

Figure 11 proves that welfare under long-term debt is greater than under short-term debt. In particular, points (1) and (2) indicate a change in the optimal government policy coming from a switch between regions. For the short-term debt case, as B increases, (1) shows the change in the government's payoff when it switches from a policy of keeping a constant level of total debt to a policy of lowering debt in T periods when $\pi > 0$. Similarly, (2) has the same interpretation for long-term debt bonds.

In the No-Default region, both government value functions start at the same value when $B = 0$. As B increases, the optimal government spending starts to decrease, making $V(B; \delta = 0)$ decline more than $V(B; \delta > 0)$. Intuitively, when total outstanding debt increases, a higher δ allocates maturing debt in several periods versus one-period bonds. As a result, when the maturing debt that the government rolls over is a portion of the total amount of debt (i.e., $\delta > 0$), the constant spending attained $\bar{g}$ is greater compared to the case where maturing debt is the total amount of debt in every period (i.e., $\delta = 0$).

In the Crisis region, we numerically obtain that the envelope of the family of V^T curves under long-term debt (built with the logic of Figure 10) is strictly greater than the envelope of curves V^T under short-term debt. In the main text, we describe that when $\delta > 0$, the government gains come from (i) the faster decline of debt that decreases the likelihood of self-fulfilling debt crises, and (ii) the lower discount of the payoff under smooth spending $u(\bar{g})$ (conditional on the sunspot not triggering default), associated to the faster policy in (i).

Figure 12 and 13 illustrate other examples of welfare improvement under a higher maturity of debt, when creditors' belief of government default increases from $\pi = 0.0001$ to $\pi = 0.001$ and $\pi = 0.01$.[13] In order to have a Crisis region where the optimal choice of T varies when $\pi = 0.01$, we changed the default penalty from $Z = 0.9$ to $Z = 0.86$. Notice also that these figures illustate the

[13]In our numerical example, a belief $\pi = 0.01$ is half of the probability of default specified in Cole and Kehoe (1996). For another specifications on π, see also Conesa and Kehoe (2017).

envelopes of the value functions for a portion of the Crisis region, specifically when $T^*(B_0) = 1$. However, when considering a greater range of B, the envelope of $\delta = 0$ switches to $T^*(B_0) = 2$ before $\delta = 0.1$ and $\delta = 0.2$ – and the same happens for $\delta = 0.1$ compared to $\delta = 0.2$.

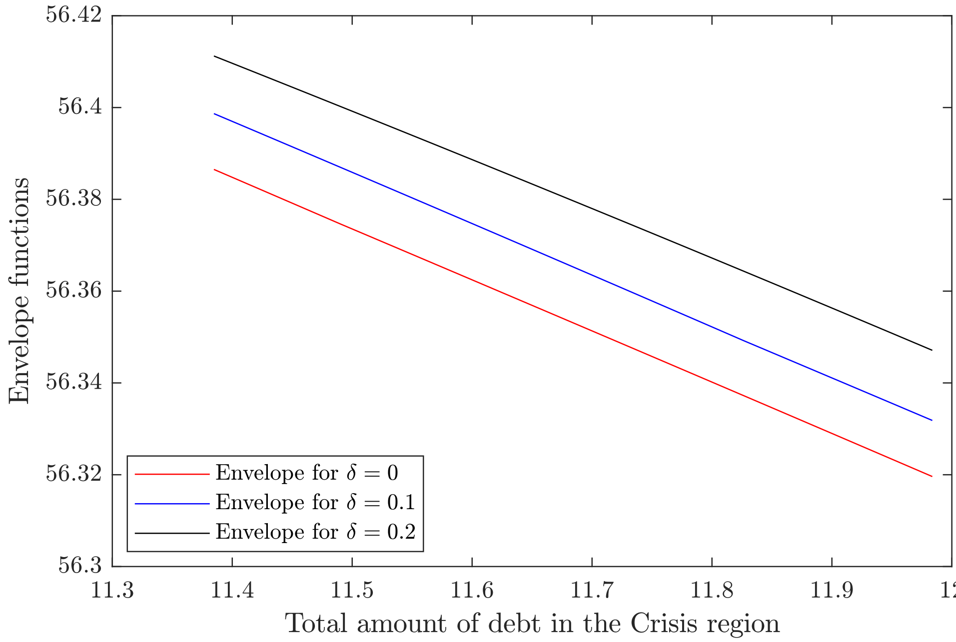

Fig. 12. Envelope functions for short-term and long-term debt, as total amount of debt varies. ($\pi = 0.001$, $\beta = 0.96$, $\delta = \{0, 0.1, 0.2\}$, $Z = 0.9$, $y = 10$ and $u(\cdot) = \ln(\cdot)$.)

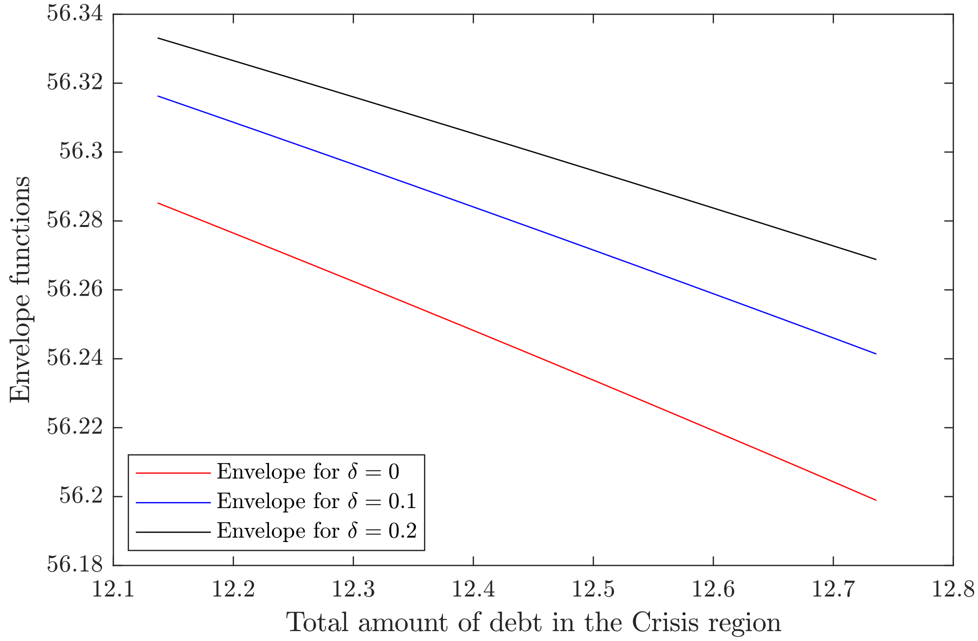

Fig. 13. Envelope functions for short-term and long-term debt, as total amount of debt varies. ($\pi = 0.01$, $\beta = 0.96$, $\delta = \{0, 0.1, 0.2\}$, $Z = 0.86$, $y = 10$ and $u(\cdot) = \ln(\cdot)$.)

As Figure 12 and 13 show, there is a welfare improvement when δ rises, showing that the findings hold when beliefs rise.