Forward Guidance Credibility and Fiscal Policy*

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Abstract

This paper studies the interaction between a discretionary central bank and a fiscal authority. The analysis focuses on repeated liquidity trap episodes requiring the central bank to rely on unconventional monetary tools, specifically, forward guidance. Confirming earlier literature, I show that forward guidance policies can be made credible using reputation built during repeated liquidity traps. The key contribution of my work is to investigate how the presence of fiscal stabilization policy affects the credibility of the Central Bank. I show that an increase in the effectiveness of fiscal stabilization policy reduces the range of credible forward guidance announcements that the central bank can implement. Finally, I show that forward guidance can crowd out fiscal effort and result in a loose monetary-tight fiscal policy mix during recessions.

Keywords: Monetary policy, Fiscal policy, Liquidity trap, Forward guidance, Reputation

JEL Codes: E52, E58, E61, E62, E63

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"[...] The Committee will maintain the target range for the federal funds rate at 0 *to* $\frac{1}{4}$ *percent and anticipates that economic conditions are likely to warrant exceptionally low levels of the federal funds rate for an extended period. [...]"*

[Minutes of the Federal Open Market Committee – March 17-18, 2009]

1 Introduction

For almost two decades now, liquidity trap episodes in OECD and Euro Area economies have constrained the ability of central banks to counter recessions with their conventional instruments. In view of such monetary policy constraint, and to mitigate the collapse in output, governments resorted to fiscal stabilization policies backed with debt. In addition, central banks also resorted to new monetary policy tools in order to regain their ability to stimulate the economy. The interaction between these new monetary policies (also known as 'unconventional') and fiscal policy is the subject of this study.

This paper focuses on one of the most popular unconventional monetary policies, known as forward guidance, and its interaction with fiscal stabilization policies. As exemplified by the FOMC announcement from March 2009 cited above, forward guidance (FG hereafter) involves disclosing intended future paths for nominal interest rates. The primary objective of such policy is to influence the expectations of the public about the future path of the economy, in turn influencing their current actions.

But like any other promise, a critical aspect of FG is that it needs to be credible, i.e., the private sector needs to believe that the central bank will follow through with its promises. To overcome this limitation and explore the effectiveness of FG, early analyses studied central banks that can commit to any announcement (often called a committed central bank). Under commitment, the central bank promises policies that the private sector always expects, and credibility concerns are eliminated.[1] Without commitment, the central bank acts under discretion, and credibility becomes important. This idea is closely related to time-inconsistency of optimal policies due to Kydland and Prescott (1977)'s seminal work.

Works that followed showed that, when the central bank is discretionary and liquidity traps occur repeatedly, reputation can help restore forward guidance credibility and, therefore, its effectiveness.[2] Away from liquidity traps, a central bank with reputational concerns has incentives to align its actions to what the private sector expects (i.e., the implementation of FG promises) if such actions improve liquidity trap outcomes relative to the outcomes

[1] In early analyses, Krugman et al. (1998) and Eggertsson and Woodford (2003) initially showed that promises of future expansionary monetary policies can improve contemporaneous consumption and inflation when a central bank can commit to such policies.

[2] See Nakata (2018), Dong and Young (2019), Nakata and Sunakawa (2019) for examples of ways to restore credibility when the central bank acts under discretion.

attained without FG. These latter outcomes without FG take place if the central bank ever deviates, in which case the private sector punishes the central bank by never believing the monetary authority's promises again. This result, however, is obtained in the environment where the central bank is the sole active policy actor in the economy. However, as previous recessions provide evidence of joint fiscal and monetary policy responses to crises, understanding the effects that the presence of an active fiscal authority may have on the credibility of central bank's forward guidance becomes important. To see why fiscal policy matters for reputation-based FG, it is straightforward to observe that, while fiscal stabilization policies can counter recessions, reputation-based FG relies on recessions to restore FG credibility. The existing literature, however, has largely been silent on this question – a gap that this paper attempts to fill in.

This paper studies how uncoordinated fiscal and monetary authorities interact during repeated liquidity traps. Specifically, I analyze whether and how one type of fiscal policy (a debt-financed fiscal expansion) can impact the credibility of reputation-based forward guidance. In addition, I explore whether this type of fiscal stabilization policy and forward guidance exhibit complementarity or substitutability during liquidity traps. To answer these questions, I modify the framework presented in Nakata (2018), who models a discretionary central bank with reputational concerns that repeatedly interacts with a competitive private sector, and then studies whether the commitment monetary policy can be sustained under trigger strategies. In this paper, I extend Nakata's analysis in two ways. First, I incorporate fiscal stabilization policy. In particular, after the central bank chooses its policy, but before the private sector responds to the central bank by forming future expectations, I introduce a fiscal agency whose response takes into account the central bank's policy that is already in place. This modification allows us to analyze how a fiscal agency can affect the ability of a discretionary central bank to fight recessions using FG.[3] Second, Nakata's study focuses on the sustainability of the optimal monetary policy. Following recent papers that look into policies that are sub-optimal relative to the commitment policy (but easier to communicate, as in Walsh, 2018), I focus on all the FG policies that the central bank can sustain.[4]

The model economy consists of three types of agents: (i) households, who are non-Ricardian due to finite planning horizon as in Devereux (2010),[5] which generates wealth

[3] Dixit and Lambertini (2003) is an often-cited paper analyzing fiscal and monetary interactions through the lens of a game. Other examples include Adam and Billi (2008), Eggertsson (2011a), Davig and Gürkaynak (2015), Gnocchi and Lambertini (2016), Camous and Matveev (2022).

[4] Walsh (2018) and Bilbiie (2019) discuss aspects of the communication of FG to the private sector.

[5] Literature analyzing the impact of fiscal stimuli packages financed with debt (like those present in past recessions) has provided microfoundations of this wealth effect. Other examples of these non-Ricardian economies when monetary policy is constrained by the zero-lower-bound (ZLB) include Smets and Trabandt (2018) and Woodford and Xie (2022). For an analysis without the ZLB, see Rigon and Zanetti (2018).

effects from debt-financed fiscal policy; (ii) a discretionary central bank, who minimizes a loss function that penalizes variations of the output gap and the inflation rate of this economy; and (iii) a fiscal authority that cares about the business cycle and tax collection costs. The central bank uses the nominal interest rate to stabilize the economy, and this policy rate is constrained by the zero lower bound (ZLB). The treasury, instead, stimulates aggregate demand via the wealth effect that private agents experience when fiscal expansions are financed with debt. Finally, the private sector structure follows Eggertsson and Woodford (2003), where liquidity trap states occur due to negative shocks to the natural interest rate (explained by a sudden rise in households' desire to save), and where the economy remains in a liquidity trap with positive probability in every period until it leaves it forever. To build reputation, I also include a state (which I call the 'economic recovery' state) where the economy transitorily exits the ZLB and, after which, the economy may return to the liquidity trap state with a fixed probability (otherwise, the economy returns to Eggertsson and Woodford's absorbing state without liquidity traps). This modification gives incentives for the central bank to care about its reputation and build credible FG using a trigger-strategy equilibrium in the spirit of Nakata (2018).

The paper consists of two parts. In the first part I characterize an equilibrium (which I call the "No-FG equilibrium") where the central bank can only use the nominal interest rate and it has no access to forward guidance during the liquidity trap state. As a result, when the central bank is constrained by the ZLB, the mitigation of the economic downturn depends solely upon fiscal stabilization policy. Indeed, I numerically show that fiscal policy can improve economic outcomes on its own, both during the liquidity trap and the recovery states. Since in this No-FG equilibrium the central bank cannot mitigate the recession with promises that affect private sector expectations, this equilibrium describes the case where the central bank lacks access to credibility. More importantly, when we explore trigger-strategies in the second part of the paper, the No-FG equilibrium will be the equilibrium towards which the economy reverts upon central bank deviations from its FG policy.

In the second part of the paper, I construct a reputation-based sustainable equilibrium with FG following Chari and Kehoe (1990).[6] Specifically, the central bank, the treasury, and the private sector will repeatedly interact, but now their past and current choices (i.e., the history of actions) will be part of the equilibrium characterization. Using trigger-strategies, the central bank cares about its reputation of implementing promised policies, and if the central bank's actions do not conform with its past promises (and, therefore, with private sector expectations), the economy falls into the No-FG equilibrium forever.[7]

[6] See, for example, Nakata (2018); Dong and Young (2019), and Nakata and Sunakawa (2019).

[7] Nakata and Sunakawa (2019) considers finite-time punishments. I do not study this scenario here.

I first show that, during recessions, forward guidance announcements can be made credible if the central bank keeps its reputation of implementing these announcements. The standard operating mechanism from the literature that makes FG credible needs the private sector to rationally anticipate that the central bank indeed prefers to implement the FG promise. Such mechanism involves the central bank balancing its losses from FG versus its optimal discretionary policy for each pair of states of the economy, respectively – the liquidity trap state and the recovery state. Intuitively, on the one hand FG creates large central bank losses in the recovery state (since the FG rate triggers larger inflation and output gap variations, relative to the higher discretionary rate) but also FG compensates these losses during a liquidity trap (since there is an aggregate demand stimulus that FG induces via private sector expectations that counters deflation and the recession at the ZLB). On the other hand, the discretionary policy involves low central bank losses during the recovery state (since the discretionary policy is not constrained by the ZLB and it is optimal) but larger losses during a liquidity trap (i.e., when the discretionary policy is constrained by the ZLB and the central bank cannot affect private sector expectations with FG). If the FG losses are smaller relative to the optimal discretionary policy losses, FG will be credible. But in this standard mechanism, fiscal policy is absent. In this paper, instead, I suggest that active fiscal policy can be relevant to this mechanism (and, therefore, to the characterization of credible FG) since fiscal policy can improve outcomes in every state of the economy (as argued in the first part of the paper). In particular, if fiscal policy can mitigate the recession during the liquidity trap state, fiscal policy can prevent the implementation of FG as it alters the costs the central bank balances when it tries to make FG credible. In this paper, I use this intuition to argue that we can still quantitatively characterize credible FG announcements even under the presence of fiscal stabilization policy – as long as fiscal policy does not fully counter the recession.

Next, I characterize a credibility region collecting which FG policies remain credible when the effectiveness of fiscal policy against recessions varies. The effectiveness is measured by the magnitude of the wealth effect that households enjoy when fiscal policy is debt-financed. When fiscal stabilization policy becomes more effective, I find that (i) it reduces the range of credible forward guidance announcements available for the central bank; and (ii) it only leaves the central bank with access to more conservative FG announcements. This can be explained by the policies at the extremes of the credibility region, namely, the least and the most expansionary FG policies. The least expansionary FG policy reflects the discretionary monetary policy followed in the No-FG equilibrium. There, the central bank is (trivially) promising the policy rate that generates the discretionary outcomes. The most expansionary FG policy, instead, reflects the policy that exhausts the credibility gains. When fiscal policy becomes more effective against recessions, it reduces the output gap and it lowers inflation in the No-FG equilibrium. But when the central bank can use reputation, this improvement in outcomes from the No-FG equilibrium implies a milder punishment that the central bank faces if it ever deviates from the FG policy. This contributes to the overall erosion of the credibility of FG (i.e., the reduction of the credibility region), and can be captured by two effects. The first effect is that a more effective fiscal policy makes recessions milder during a liquidity trap state. As a result, it is more difficult for the central bank to sustain large FG policies using reputation and, therefore, the most expansionary FG policies available to the central bank drop out from the credibility region – i.e., the most expansionary policy rate from the credibility region rises. The second effect of more effective fiscal policy is that it also mitigates the economy's bad outcomes during the recovery state. This results in a rise in the least expansionary policy rate from the credibility region. This is because in the recovery state, more effective fiscal policy gives the central bank more leeway to set nominal interest rates closer to the natural interest rate of the economy. These two effects together explain that the credible FG levels available to the central bank become more conservative, as fiscal policy increases both extremes of the credibility region. Finally, since the rising of the extremes of the credibility region gradually approaches them to the natural interest rate of the economy (which is positive in the recovery state, and which the central bank tries to fully absorb) the range of FG policies available to the central bank must shrink.

Finally, I conduct a policy experiment to compare the equilibrium outcomes that result when we transition from the No-FG equilibrium to one where a central bank has access to forward guidance. I find that the central bank's ability to conduct forward guidance can induce the treasury to conduct less expansionary fiscal policies compared to the case without forward guidance. In particular, the salient aspect of this result is that a loose monetary-tight fiscal policy mix can emerge in equilibrium during recessions. This result is of relevance as portrayed by some macro developments from past recessions. On June, September and December 2013, the US' Federal Open Market Committee repeatedly issued statements discussing that fiscal policy was 'restraining economic growth', and signalling monetary action in the form of low rates for a considerable period of time.[8,9] The model in this paper formally captures this narrative evidence through the expectations mechanism present in the policy responses from the central bank and the treasury. Intuitively, since forward guidance involves promises of future outcomes, then during a liquidity trap credible FG may

[8]For more details about this, see Davig and Gürkaynak (2015).

[9]https://www.federalreserve.gov/newsevents/pressreleases/monetary20130619a.htm,
https://www.federalreserve.gov/newsevents/pressreleases/monetary20130918a.htm,
https://www.federalreserve.gov/newsevents/pressreleases/monetary20131218a.htm (Accessed on November 6, 2022.)

crowd out fiscal effort (relative to the case with no FG) if the best response function of fiscal policy depends on those future variables that FG affects, too. This result can highlight the importance that portraying fiscal and monetary interactions from a game-theoretic approach has, as it provides insights of cases where substitutability between fiscal policy and FG can emerge even during recessions.

Related literature. Over the past two decades a growing body of work has analyzed alternative stabilization policies when the nominal interest rate hits the ZLB. On the one hand, a branch of that literature has studied whether forward guidance can mitigate crises in economies subject to liquidity traps.[10] On the other hand, a large set of papers has analyzed whether fiscal policy can be a substitute stabilization tool when the monetary authority is assumed to only have access to its standard policy instrument (but which is assumed to be exhausted, namely, at the ZLB). As will become clear shortly, this paper seeks to build a linkage between these two strands.

The first branch related to forward guidance has highlighted that promises of monetary expansions can help increase consumption and fight deflation when the public rationally expects the central bank to deliver loose monetary policies in the future. Stemming from the succinct model in the seminal paper of Krugman et al. (1998), this result has been articulated in more general dynamic stochastic general equilibrium settings where the economy starts in a liquidity trap and it can escape from it with some probability. Eggertsson and Woodford (2003), in particular, has became one of the workhorse models in this literature characterizing optimal monetary policies in discrete time, and it is also the setup I use in this paper.[11] Forward guidance involves making promises of future paths of nominal interest rates, and in this context Eggertsson and Woodford (2003) show that when the central bank has no credibility issues (i.e., it can credibly commit to any policy it announces, often called the Ramsey policy), monetary policy promises effectively affect macro variables by shaping private sector expectations. Since then, several applications with similar setups to that in Eggertsson and Woodford have emerged analyzing perfect foresight (Jung et al., 2005), recurrent liquidity traps episodes (Adam and Billi, 2006; Nakov, 2008), continuous time (Werning, 2012), to name a few. But the commitment assumption had previously been known to be critical by the literature (see Kydland and Prescott, 1977, for a seminal exposition on time-consistency, and Clarida et al., 1999, for an early application to monetary policy).[12] A main

[10] There are also other monetary policy instruments used by Central Banks to circumvent the restrictions imposed by the ZLB (e.g., quantitative easing). See Woodford (2012) for an account of these alternative policies.

[11] Jung et al. (2005) do this in a deterministic setup. For other studies using this model, see for example Nakov (2008), Eggertsson (2006), and Christiano et al. (2011).

[12] Even absent the ZLB constraint, Clarida et al. (1999) showed how central bank's inability to commit

lesson is that the commitment assumption is time-inconsistent (i.e., past promises do not survive revisions when the future becomes the present) and, in particular, in the context of models with a ZLB constraint, lack of commitment (also known as central bank discretion) can render monetary policy ineffective when the central bank cannot convince the public that it will implement its promised monetary policies. To circumvent the lack of credibility that is intrinsic to discretionary policymakers, the framework introduced by Chari and Kehoe (1990) has been a popular way in the macroeconomics literature to conceive credible forward guidance policies with credibility based on reputation and where such policies can potentially lead to outcomes better than those under discretion. Featuring an environment with repeated interactions between a non-committed policymaker and infinitely many competitive private agents, policies are deemed credible as long as the policymaker prefers them rather than deviating but being punished with a reversion to the discretionary outcome. Chari and Kehoe's reputational equilibrium concept (called sustainable equilibrium, or simply SE) was early implemented by Kurozumi (2008) to analyze under what circumstances the Ramsey monetary policy can be made credible in general setups without the ZLB.[13] This analysis was later extended to economies where the ZLB is present (Nakata, 2018) to analyze whether forward guidance following Ramsey paths can be made credible. Recent contributions continued investigating the credibility of reputation-based FG analyzing how the length of the punishment periods affects credibility of reputation-based FG (Nakata and Sunakawa, 2019), and what the full set of sustainable FG equilibria is when the punishment is endogenously determined (Dong and Young, 2019). This body of work has modelled how forward guidance can become credible and emerge in equilibrium in discretionary setups, and also analyzed the sensitivity of forward guidance to key parameters of the economy. But although liquidity trap episodes have been accompanied by expansionary fiscal policies in the 2008 world recession, this track of the literature typically abstracts from modelling fiscal policy, thus leaving unexplored any potential linkages between fiscal policy and reputation-based forward guidance.[14] Such linkage is relevant to understand whether fiscal policy may erode the credibility of forward guidance or not, and it is the step taken in this paper.

A second branch of the literature analyzing effective macro policies during liquidity traps has focused on fiscal policy as a stabilization tool. A popular modelling approach within this branch is the so-called Ricardian setup, based on the impact of expansionary fiscal policies

and, therefore, affect private sector expectations matters when monetary policy cannot smooth out the impact of a shock. This assumption also highlights the long discussed time-inconsistency problem in the choice of monetary policy documented by Kydland and Prescott (1977).

[13] Chari et al. (1998) is an early use of the sustainable equilibrium apparatus showing how the Ramsey outcome can be achieved when the economy faces expectation traps. See also Ireland (1997).

[14] Recent studies also analyze credible FG outcomes that are sub-optimal relative to Ramsey (but better than those under discretion (Walsh, 2018) and the optimal duration of FG (Bilbiie, 2019).

at the ZLB while assuming away any impact of budget deficits as a means of financing such policies. Examples of such applications have shown that fiscal policy is effective at the ZLB by analyzing the multipliers of distortionary and lump-sum taxation, as well as government spending (see for example Christiano et al., 2011; Eggertsson, 2011b; Woodford, 2011). But one departure from the deficit-irrelevance approach described before has been presented in the growing literature investigating the role of debt-financed fiscal policies from a non-Ricardian perspective. This approach has been largely motivated by the observed developments in fiscal policy practice during the Great Financial Crisis. In particular, the large fiscal stimulus packages backed with debt[15] observed during the financial crisis of 2008 triggered investigations about the effect of debt financing in non-Ricardian models featuring liquidity traps. Such studies revisited the effect of debt-financed spending on fiscal multipliers (Devereux, 2010), how optimal government spending responds to different tax instruments (Nakata, 2017), and the impact of a risk premium on government debt (Smets and Trabandt, 2018), to name a few. In this respect, the presence of debt in non-Ricardian environments contributes to new margins of gains coming from fiscal policy, either because of the wealth effect that debt creates when households are finitely lived (Devereux, 2010; Smets and Trabandt, 2018), or because the discrepancy between initial and steady-state debt levels provide incentives for fiscal policy to introduce gains when the nominal interest rate hits the ZLB (Nakata, 2017). This strand of the literature can show that active fiscal policy is likely effective to mitigate recessions during liquidity traps, but a simplifying assumption in this class of models is to usually summarize monetary policy with a fixed rule (e.g., a Taylor rule), or with a committed central bank (implementing the Ramsey policy).

The explorations carried out by both strands of the literature yield several lessons regarding the response of economic outcomes to stabilization policies. But it is important to stress that during the 2008 financial crisis neither central banks, nor treasuries acted in isolation: An interplay between fiscal and monetary policy was observed as both agencies can alleviate recessions. In this respect, forward guidance has received attention from multiple dimensions but, to the best of my knowledge, in the forward guidance literature it has not been carried out an analysis of whether credibility can be affected or not by the existence of another big player of the economy – here, the treasury.[16] Therefore, this work attempts to contribute to the literature on credible forward guidance by undertaking a modest step in merging this

[15]For a textbook exposition with a survey of fiscal policies, see Garın et al. (2018). For the management of debt during the 2008 financial crisis, see Greenwood et al. (2014).

[16]In this context, a recent study linking fiscal and monetary policy responses during liquidity traps belongs to Woodford and Xie (2022), which also considers a non-Ricardian environment and it discusses the stabilization power of forward guidance and countercyclical fiscal transfers. However, the focus in my paper is different in that I study how fiscal policy can affect the credibility of forward guidance.

strand of the literature with the strand focused on the conduct of fiscal stabilization policies during liquidity traps, in order to explore whether FG credibility can be eroded by fiscal action.

Finally, since here I present a setup with two government agencies, a central bank and a treasury, this paper also relates to the literature that analyzes how the interaction between monetary and fiscal authorities can shape macroeconomic outcomes. In this field, the institutional arrangement that eventually guides the conduct of monetary and fiscal policy is relevant. This literature can be traced back to the early analyses of Sargent et al. (1981), also addressed in the active- and passive-policy description present in Leeper (1991), where inflation under fiscal-dominance may be different to that under monetary dominance. More recently, a subset of papers within this literature (Dixit and Lambertini, 2003; Adam and Billi, 2008) has further investigated these interactions considering different arrangements between strategic monetary and fiscal policymakers. These studies have provided a comparative study of the macro implications of different timing protocols between policymakers – e.g., simultaneous and leader-follower sequences of moves. Given that there is no strong consensus about the correct sequence of moves, different studies usually fix some timing-protocol to draw different fiscal and monetary policy lessons, such as simultaneous games to analyze the limitations of monetary policy (Davig and Gürkaynak, 2015) or fiscal multipliers (Eggertsson, 2011a); central bank leadership to study welfare gains from monetary commitment when a central bank moves before a discretionary treasury (Gnocchi and Lambertini, 2016); the design of monetary rules comprehensive of treasury deviations (Camous and Matveev, 2022), among others. In this respect, this paper uses the framework of central bank and treasury interaction to analyze joint fiscal and monetary policy behavior during recessions, and it considers a central bank leading the treasury for practical purposes as it simplifies setting up sustainable equilibria. The paper contributes to this literature of interactive fiscal and monetary policymakers (and its impact on macro outcomes) by addressing the macro impact of monetary policy (in the form of forward guidance) and fiscal policy (expansionary, with debt-financing wealth effects). Specifically, one qualitatively insight of this analysis is to explore whether these policies respond as substitutes or complements. To the best of my knowledge, it has not been studied how the interaction between fiscal and monetary policymakers can affect central bank's forward guidance from a substitutability/complementarity perspective (when FG is based on reputation), and therefore this paper takes a step in this direction to analyze some of the limitations that reputation-based forward guidance may face in the presence of active fiscal policy.

The remainder of the paper is organized as follows. Section 2 describes the model setup, and Section 3 outlines the equilibrium when the central bank has no access to forward guidance policies. Section 4 then presents the equilibrium when the central bank can conduct forward guidance and introduces the key results. Section 5 concludes.

2 The Model

In this section I present an economy consisting of a Private Sector, a central bank and a treasury. The description of the Private Sector uses log-linear approximations of the equilibrium conditions of the standard New Keynesian model,[17] augmented to capture the impact of debt-financed fiscal policy. The structure of the Private Sector closely resembles those of Kirsanova et al. (2005), Devereux (2010) and Rigon and Zanetti (2018), which incorporate debt in the dynamic IS equation. The description of the central bank and the treasury uses a linear-quadratic setup stemming from the standard models of Barro (1979), Barro and Gordon (1983) and Rogoff (1985), which are still widely used in the macro literature to describe the preferences of monetary and fiscal authorities (see, for example, Clarida et al. (1999), Dixit and Lambertini (2003), Jung et al. (2005), Eggertsson (2011a), Davig and Gürkaynak (2015), Camous and Matveev (2019)). There is an explicit zero-lower-bound (ZLB) constraint on nominal interest rates (the monetary policy instrument), and liquidity traps are described in the tradition of the widely used approach of Eggertsson and Woodford (2003), modified to incorporate repeated liquidity traps episodes that allow building reputation. In what follows, I present the building blocks of the model, and then turn to a detailed discussion of the assumptions before presenting the equilibrium outcomes.

2.1 Private sector

The environment summarizes a Private Sector (thereafter PS) with two equilibrium conditions: A New Keynesian Phillips Curve (NKPC) and a Dynamic Investment-Savings equation (DIS). For every period $t \geq 0$, the NKPC and DIS are

$$\pi_t = \kappa y_t + \beta E_t \pi_{t+1}, \tag{1}$$

and

$$y_t = E_t y_{t+1} - (i_t - E_t \pi_{t+1} - r_t) + \gamma b_t. \tag{2}$$

Operator E_t denotes rational expectations of the Private Sector over variables in $t+1$ given information from period t, parameters κ, γ and β are positive constants, and variable r_t

[17]See, for example, Eggertsson and Woodford (2003). For a textbook presentation of the equilibrium conditions, the reader is referred to Woodford (2003) and Galí (2008).

is the exogenously given natural interest rate, or the shock for short – described later in this section. Variable π_t is the inflation rate, namely, the log-deviation of inflation from its steady state. Likewise, variable y_t represents the output gap capturing the log deviation of output from its steady state level. Finally, variables i_t and b_t capture the impact of monetary and fiscal policy, respectively: Variable i_t is the central bank's monetary policy instrument representing the nominal interest rate, and variable b_t represents real debt-to-GDP ratio fluctuations around an initial debt-to-GDP ratio – set at 0 for simplicity; see Devereux (2010) for a similar treatment. Equations (1) and (2) characterize the optimal behavior of firms and households. As it is common in the literature (see, for instance, the textbook treatment in Woodford, 2003, pp. 405, 406), I assume that the log-linear approximations of these equilibrium conditions have been performed around a zero steady state level of inflation. To facilitate the exposition, I further assume that there is no discrepancy between steady state output and the efficient level of output, and I shall state here (and argue later) that the steady state level of output gap is zero. Finally, it is worth to point out that the nominal interest rate is not defined in deviation terms, so that the zero lower bound constraint (to be displayed later) only requires i_t to be nonnegative.

Equation (1) shows that current period inflation is increasing in the output gap in proportion κ (the slope of the Phillips Curve), as well as in next period's expectations over inflation (discounted by factor $\beta \in (0,1)$) due to forward looking firms facing price rigidities. Equation (2) expresses the log-linear version of the Euler equation relating households' optimal present and future consumption choices as a function of expected inflation, the exogenous shock, and fiscal and monetary policies. Since market clearing holds, I adopt the convention of interpreting the DIS curve in output gap terms instead of consumption. Eq. (2) then shows that the nominal interest rate, i_t, and debt financed fiscal policy, b_t, can stabilize the output gap by absorbing (i) the shock to the natural interest rate (r_t), and (ii) changes in inflation and output gap expectations ($E_t\pi_{t+1}$ and $E_t y_{t+1}$). Variable b_t captures the fluctuation in aggregate real bonds held by households, which correspond to treasury's debt from period t (to be repaid in $t+1$), and issued to finance time-t fiscal policy (described below). For this reason, in what follows I will also refer to variable b_t as debt-financed fiscal policy – the rationale for b_t in the DIS is relegated to section 2.4.

2.2 Monetary and fiscal authorities

The economy features two government agencies: a monetary authority and a fiscal authority. Along these lines, I will also refer to these agencies as the central bank and the treasury, or simply using labels CB (for central bank) and Tr (for treasury), respectively.

The central bank is concerned with rates of inflation and output gaps. Hence, at period t, the central bank preferences are

$$L_t^{CB} \equiv E_t \sum_{k=0}^{\infty} \beta^k \left(l_{t+k}^{CB} \left(y_{t+k}, \pi_{t+k} \right) \right), \tag{3}$$

where l_{t+k}^{CB} denotes the central bank's per-period loss function,

$$l_{t+k}^{CB} \equiv -\frac{1}{2} \left(\alpha_y^{CB} y_{t+k}^2 + \alpha_\pi^{CB} \pi_{t+k}^2 \right), \tag{4}$$

and L_t^{CB} is the expected present discounted value of the central bank's losses, where parameter $\beta \in (0, 1)$ is the central bank's discount factor, and expectations are conditional on the information available in period t. I further assume that the expectations operator and the discount factor of the central bank coincide with those of the Private Sector – below I shall assume the same for the treasury. It is important to point out that one implication of Eq. (4) is that the central bank's target level of inflation coincides with the economy's steady state level of inflation. Furthermore, the central bank has a zero targeted level of the output gap, which will match with the steady state level of the output gap. This also rules out surprise inflation – see Jung et al., 2005 for a detailed discussion. Finally, positive parameters α_π^{CB} and α_y^{CB} describe a central bank that prefers price stability and dislikes output gap distortions, respectively. In line with other works in the literature (e.g., Adam and Billi, 2008), in this setup I assume that the central bank cares more about inflation stabilization than output gap stabilization, namely, $\alpha_\pi^{CB} > \alpha_y^{CB}$.

As stated above, the central bank uses the nominal interest rate, i_t, to absorb shocks to the natural interest rate, and it features a zero-lower bound constraint,

$$i_t \geq 0 \quad \forall t. \tag{5}$$

In addition to the central bank, there is a treasury that controls fiscal instruments that can mitigate recessions. The fiscal authority has access to debt, b_t, as defined before, and it performs lump-sum transfers to (it collects lump-sum taxes from) currently alive households. Specifically, variables ℓ_t and τ_t denote the real lump-sum-transfers-to-GDP and real tax-collections-to-GDP ratios relative to initial ratio levels, respectively – both initial ratio levels set at zero for simplicity. The economy then features a treasury with expected present discounted value of its per-period losses at time t given by

$$L_t^{Tr} \equiv E_t \sum_{k=0}^{\infty} \beta^k \left(l_{t+k}^{Tr} \left(y_{t+k}, \tau_{t+k} \right) \right), \tag{6}$$

with per-period losses l_{t+k}^{Tr} defined by

$$l_{t+k}^{Tr} \equiv -\frac{1}{2} \left(\alpha_y^{Tr} y_{t+k}^2 + \alpha_\tau^{Tr} \tau_{t+k}^2 \right). \tag{7}$$

with positive parameters α_y^{Tr} and α_τ^{Tr}, and where the treasury's discount factor β and expectations E_t coincide with those of the central bank and the Private Sector. Eq. (7) indicates that the treasury wants to minimize output gap distortions (term $\alpha_y^{Tr} y_{t+k}^2$) and tax collection costs (term $\alpha_\tau^{Tr} \tau_{t+k}^2$).

The treasury's intertemporal budget constraint at time t is

$$\ell_t + (1+r)\, b_{t-1} = b_t + \tau_t. \tag{8}$$

This specification represents a linear approximation of a standard treasury budget constraint in the context of a New Keynesian model. The right-hand-side variables represent the sources of income for the treasury: The treasury issues b_t (time-t) units of one-period real bond holdings that pay $(1+r)$ per unit at time $t+1$, with r the steady state level of the natural interest rate,[18] and it collects lump-sum taxes (or taxes, for short), τ_t . Conversely, the left-hand-side variables in Eq. (8) represent treasury expenditures: Payments of outstanding debt from period $t-1$ (carried up to period t and yielding interest $(1+r)$), and lump-sum transfers to the private sector, ℓ_t. Note that I abstract from seigniorage revenue transfers from the central bank to the treasury. This is because I want to keep track of the simplest exposition of the model where forward guidance only impacts the economy's variables via PS expectations. For a brief discussion about the absence of seigniorage revenues in this type of models, see Eggertsson (2006).

The treasury also faces two additional constraints on bonds, lump-sum transfers and taxes. There is a finitely large (exogenous) limit to the total debt the treasury can borrow to finance expansionary fiscal policy given by

$$0 \leq b_t \leq \bar{b} < \infty, \quad \forall t, \tag{9}$$

[18] Debt from past periods yields the steady state value of r_t, namely r, when the linear approximation of the Treasury's budget constraint is performed around an initial steady state with a debt–to-GDP ratio equal to zero – for example, see Devereux (2010).

and there is also a non-negativity constraint on lump-sum transfers and lump sum taxes,

$$\ell_t \geq 0, \quad \forall t, \tag{10}$$

$$\tau_t \geq 0, \quad \forall t. \tag{11}$$

These restrictions ensure that the treasury always transfers (or collects) positive or zero amounts to the (from the) private sector.

2.3 Exogenous shock and competitive equilibrium

Consider a state of the world, s_t, governed by a Markov chain. States take values from the set $\mathcal{S} = \{Z, R, S\}$. State Z is the *zero lower bound* state, or ZLB, and is alternatively called a Liquidity Trap state or a crisis state. States R and S are the Economic Recovery state (or just Recovery state) and the Steady-State, respectively – refer to section 2.4 for an explanation of this three-state structure. Associated to each state of the world there is a value that the natural interest rate, r_t, can take. Specifically, r_t maps states of the world into two values: A negative natural interest rate, $\underline{r} < 0$, or the positive steady-state natural interest rate, $r > 0$. When the state s_t is R or S, then r_t becomes $r_t(R) = r_t(S) = r$. For simplicity I shall denote $r_t(R) \equiv r_R (= r)$ and $r_t(S) \equiv r_S (= r)$. This notation will be helpful to avoid ambiguities and identify the state of the variables in the optimization problems of the central bank and the treasury. However, I will simplify notation $\{r_R, r_S\}$ with r when the state is no longer relevant. In addition, in this model the desire to save from agents can suddenly increase, which is captured by the exogenous shock to the natural interest rate. Formally, when state $s_t = Z$, then $r_t(Z) \equiv r_Z = \underline{r} < 0$. The rationale behind this shock to the natural interest rate could be seen as a negative demand shock – which is a common structure in the literature; see, for example, Eggertsson (2006, 2011b). As I shall argue in the solution to the central bank problem in state Z, I shall further assume that $\underline{r}$ is negative enough to make the ZLB constraint on nominal interest rates bind in equilibrium.

The transition between states of the economy is the following. If the economy is in state Z, it moves to the Recovery state R with probability 1. If the economy is in state R, it transitions to state Z with positive probability $p \in (0, 1)$ (i.e., $p = \Pr(s_{t+1} = Z | \, s_t = R)$), and it arrives state S with complementary probability, $1 - p$ (i.e., $1 - p = \Pr(s_{t+1} = S | \, s_t = R)$). Finally, if the economy is in state S, it remains in that state with probability 1.

Figure 1 shows all possible histories starting at time 0 and until period 5 for this three-state Markov chain. The upper-most branch shows the case where the economy reverts to the (absorbing) steady state at time $t = 2$ with probability $1 - p$, right after the first

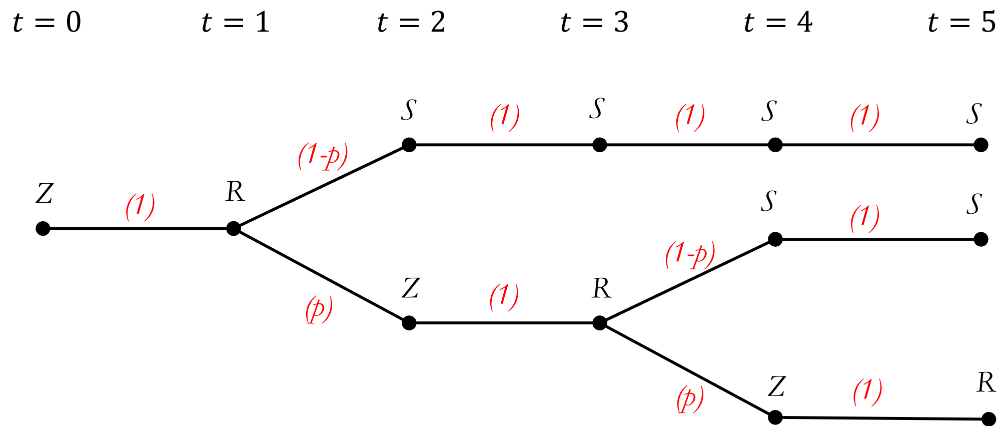


Figure 1: All possible histories of states of the world that can be reached at $t = 5$ (starting from $t = 0$). Transition probabilities as per matrix P are shown in parenthesis in red.

liquidity trap episode. With complementary probability p, the economy falls into a liquidity trap again at $t = 2$, period after which the economy may return to the absorbing steady state in period $t = 4$ (or the ZLB episode may take place again, and this cycle repeats).

Having described the states of the economy, I revisit the treasury's constraints to make the following simplifying assumptions. When the economy is in a liquidity trap (state Z), we assume that the treasury can only use lump-sum transfers (instead of lump-sum tax rebates) to mitigate recessions. The impact of this policy will be reflected in the DIS equation, where lump-sum transfers financed with debt have a positive impact measured by γb_t. On the contrary, when the economy recovers, or when it reaches its steady state (states R, S), the treasury retires debt by collecting taxes, and leaves the central bank regain control of stabilizing the economy by adjusting its nominal interest rate. Therefore, we add the following equations,

$$b_t \times \mathbf{1}_{\{r \neq \underline{r}\}} = 0, \qquad \ell_t \times \mathbf{1}_{\{r \neq \underline{r}\}} = 0, \qquad \tau_t \times \mathbf{1}_{\{r = \underline{r}\}} = 0, \quad \forall t. \tag{12}$$

The role of the indicator functions will be to identify the fiscal instruments that the treasury is allowed to use in each state. For instance, the indicator variable prevents ℓ_t and τ_t from being simultaneously positive in Z states. Also, indicator functions will prevent debt from being positive in the recovery state or the steady state – a rationale for these assumptions is provided in section 2.4.

We are now ready to define a competitive equilibrium for the model economy. The following equilibrium definition is standard (see, for example, Eggertsson, 2006, 2011b and Nakata, 2018), and it presents the conditions that the private sector must satisfy in equilibrium, for any state and given any fiscal and monetary policy choices.

Definition 1 (Private sector competitive equilibrium - PSCE) *Given a stochastic process for the natural interest rate, $\{r_t\}_{t=0}^{\infty}$, an initial debt level b_0 and monetary and fiscal policies $\{i_t\}_{t=0}^{\infty}$ and $\{\tau_t, b_t, \ell_t\}_{t=0}^{\infty}$, a private sector competitive equilibrium (PSCE) is a set of inflation levels $\{\pi_t\}_{t=0}^{\infty}$ and output gaps $\{y_t\}_{t=0}^{\infty}$ that satisfy the NKPC (Eq. (1)), the DIS (Eq. (2)), the zero-lower bound constraint (5), together with the treasury's budget constraint (Eq. 8) plus the additional restrictions and assumptions on fiscal variables (Eqs. (9), (10) and (12)).*

2.4 Discussion of assumptions

Our goal is to summarize key aspects of fiscal and monetary policy interactions observed during liquidity traps; specifically, to show how fiscal policy matters for the credibility of forward guidance. To that end, the model presented in this paper needs to allow equilibrium characterizations of forward guidance based on reputation while describing an economy that faces a liquidity trap. I discuss below key simplifying assumptions that will let us keep the model tractable and obtain solutions in closed form.

Wealth effect of debt. Parameter γ is the coefficient on government debt in the DIS curve. Therefore, term γb_t shows how a treasury that implements expansionary fiscal policies financed with real units of debt affects the output gap, y_t. Term γb_t is characteristic of non-Ricardian setups that capture a wealth effect from expansionary fiscal policies, and the rationale behind it can be explained by the assumption that households face a survival probability in each period. Specifically, those that die after receiving fiscal aid financed with debt seize a wealth effect of magnitude γb_t – as they will not pay back the fiscal aid they received.[19] Term γb_t matters to our analysis as it portrays the impact of deficit-financed fiscal policies. In the context of this paper, a positive γ will be referred to as a case where fiscal policy is 'active', and γ will be analytically relevant as it allows us to control the channel via which fiscal policy affects monetary policy. (In the limiting case where γ is zero, the economy reverts to standard New Keynesian textbook representations; see, for instance, Galí, 2008).

Preferences of the fiscal authority. Similar treasury's preferences portrayed in Eq. (7) have been adopted in other studies (see, for instance, Dixit and Lambertini, 2003; and Adam and Billi, 2008), where some form of social loss function from the Private Sector is usually assigned to the fiscal authority. Before proceeding, a few further comments on Eq.

[19] See, for example, Kirsanova et al. (2005) for another application. For an analysis on how this wealth effect of debt-financed fiscal policy can formally emerge in the context of a New Keynesian model, see for example Devereux (2010), Smets and Trabandt (2018), and Rigon and Zanetti (2018), who present a microfounded model under the violation of the representative agent assumption.

(7) are in order. First, and following the standard specification of monetary policy in charge of an independent central bank (see, for example, the large literature after Rogoff, 1985), the fiscal authority does not exhibit concerns over price stability. Second, the treasury faces tax collection costs as measured by the second term in Eq. (7). The presence of this term in the loss function comes from the long tradition of modelling a fiscal authority that faces tax collection costs (see, for example, Barro, 1979; a microfounded version can be found in Eggertsson, 2011a; and a model with a similar preference specification to this paper is presented in Davig and Gürkaynak, 2015). However, note also that this term indicates that the treasury faces asymmetric costs to τ_t relative to ℓ_t: Specifically, the fiscal authority enjoys zero costs from giving out money via lump-sum transfers, but it faces a cost from collecting money back from households in the form of taxes. The rationale of this term is to capture the differential impact that giving out money relative to collecting money can have within the population of currently alive cohorts.

Central bank independence. This paper presents a central bank independent from the treasury. This assumption about separate government agencies has had a large tradition in the literature (for example, Rogoff, 1985; and for recent contributions see Dixit and Lambertini, 2003; Adam and Billi, 2008; Basso, 2009; Gnocchi and Lambertini, 2016; and Camous and Matveev, 2022). This assumption can be easily mapped into the institutional arrangements observed in many OECD and Euro Area economies from the early 2000's onward, where monetary policy is delegated to an independent central bank, and which therefore approximates the institutional arrangement that was in place when liquidity traps started to be observed. In the context of this model, the formal version of the independent central bank assumption adopted here closely follows Eggertsson (2011a), where the central bank (1) has a loss function different from the treasury losses; and (2) chooses monetary policy independently from the treasury's budget constraint and any other restrictions on fiscal variables.

Exogenous states. The Markov process is based on the popular simple stochastic model described in Eggertsson and Woodford (2003), where the shock to r_t follows a Poisson process and that it vanishes at a stochastic time where the economy reverts to an absorbing zero inflation steady state. However, in this paper I depart from their exposition in two dimensions. First, I consider one-period liquidity traps. This allows me to have debt that does not build up indefinitely – thus ruling out the possibility of exploding debt levels while, at the same time, simplifying the model exposition. Second, I implement an intermediate state following the liquidity trap where the shock vanishes, but that it is transitory – unlike the absorbing deterministic steady state in Eggertsson and Woodford (2003). This transitory intermediate state allows me to have a stochastic model in an identical fashion to Nakov

(2008) and Adam and Billi (2008) (where ZLB episodes are recurrent), but it is still simple enough to allow us build a reputational equilibrium as in Nakata (2018) and Walsh (2018) while keeping the ability to compute results analytically.[20] A sequence of Z and R states, however, may not be sufficient to characterize an equilibrium with liquidity traps. As Nakata (2018) points out, we may reach some types of equilibria where the ZLB constraint binds in every state when the frequency of the negative shock is too high. The intuition behind this unwanted outcome is straightforward. If after a state without a liquidity trap it follows a liquidity trap state, the prospect of deflation in the latter state triggers deflation in the former state (due to the forward-looking NKPC). But deflation in that no-liquidity trap state calls for lower rates and, in some cases, if deflation in the future is very likely (p is high in the context of our model), then the ZLB might be binding even away from the liquidity trap. To circumvent this issue, some options are available, and each carry their own costs. A first option is to make the non-liquidity-trap state persistent, namely, to make state R display a stochastic duration. But this comes at the cost of an additional parameter that needs to be monitored to assure the model does not exhibit liquidity traps when there is no shock – i.e., liquidity traps generated when deflation feeds itself via expectations and absent a shock, as we argued before. Another option to avoid that the ZLB binds in a non-liquidity-trap state is to keep the duration of state R fixed and append a reversion to a steady state without inflation. In this way, Private Sector's expectations are anchored to a state that disciplines deflation when the economy is away from liquidity traps to be zero (i.e., nonnegative). Although this trick adds an additional state, it largely simplifies the model in several dimensions. First, it allows us to think of the economy in the similar Poisson process from Eggertsson and Woodford (2003), where each pair of consecutive Z and R states are our repeated tuples. As such, it displays a very simple model, but with a small departure from Eggertsson and Woodford (2003) that ensures the existence of a reputational equilibrium. Second, it focuses our analysis to one-period FG, and disregard considerations that are not part of this paper like the duration of FG. And third, it keeps the model tractable as it summarizes in parameter p all the relevant information that has to be monitored to confirm that our equilibrium features a liquidity trap only in state Z.

[20] As it will be seen later, the shock is necessary because if, on the contrary, the state without the shock is absorbing, then reputation cannot be sustained – although the liquidity trap is of stochastic duration, reputation is ruled out if the ZLB is a one-time event ocurring that never repeats when it is over.

3 Scenario with no forward guidance

The objective of this section is to build an equilibrium when the central bank only solves its problem under discretion and it has no access to forward guidance. I will interchangeably refer to this problem as the discretionary problem, or the problem without forward guidance. The use of this equilibrium is twofold: First, it will allows us build the reputational equilibrium from Section 4; and second, it will serve as a benchmark for comparison when we analyze the effects of credible forward guidance on the economy's outcomes relative to the no-forward-guidance equilibrium. Hence, I first describe the policy problems given the timing of events within each period, and then I introduce the equilibrium definition and characterize optimal monetary and fiscal policies.

3.1 Equilibrium definition

Before presenting the equilibrium outcomes of this economy, I need to establish the within-period timing protocol. In the literature that characterizes sustainable policies with a discretionary policymaker (Kurozumi, 2008, Walsh, 2018, Nakata and Sunakawa, 2019), the implemented timing structure usually assumes a policymaker moving first and internalizing PS moves as in Chari and Kehoe (1990). In this model, however, there are two policymakers and, consequently, the timing protocol admits different arrangements where monetary-fiscal interaction is described with a simultaneous (Nash) or a leader-follower (Stackelberg) setup. With the final goal of characterizing how the credibility of forward guidance responds to fiscal policy parameters, the focus of this paper is, primarily, on FG as an effective stabilization tool for the monetary authority. Driven by practical purposes, I shall assume a leading monetary authority and a follower fiscal authority. The leader-follower structure simplifies the computation of deviations during the proof of forward guidance as a sustainable equilibrium.[21]

In the presence of a monetary leadership structure, the following events take place in every period $t \geq 0$. At the beginning of the period, the shock is realized. After the shock, the Central bank, the treasury and the Private Sector make their decisions in the following order. First, the Central bank makes its decision after observing the state and sets the nominal interest rate. Then, having observed both the shock and the monetary policy in place, the treasury follows and chooses fiscal policy. Finally, with these policies in place (together with the observed shock) the private sector satisfies the DIS and the NKPC by

[21] For a detailed description about the timing of events in fiscal-monetary policy setups, see Dixit and Lambertini (2003) and Adam and Billi (2008). Recent applications (absent the ZLB) also feature a leading central bank (Gnocchi and Lambertini (2016); Camous and Matveev (2022)), and cases of two strategic policymakers (Basso (2009)).

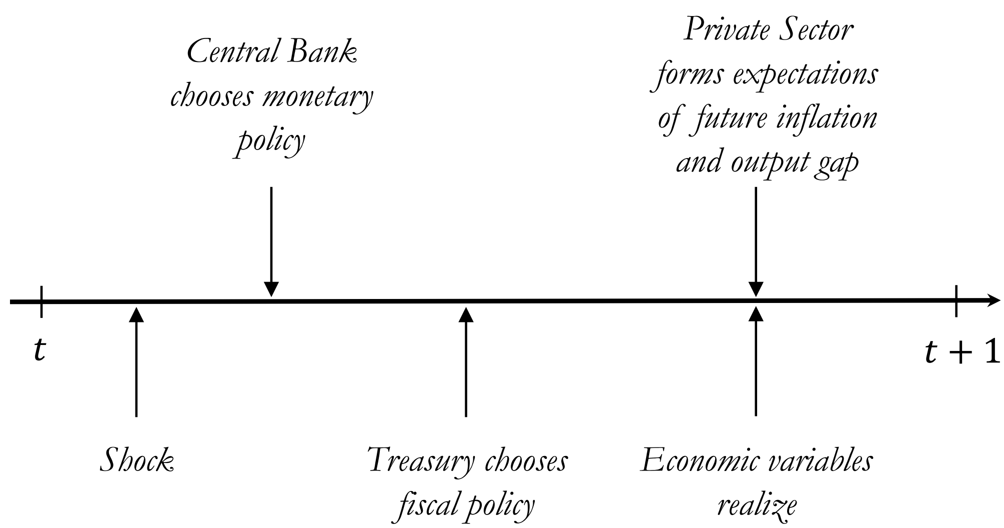


Figure 2: Sequence of events.

determining output and inflation while it forms rational expectations. The description of the events is summarized in Figure 2.

After the shock to the natural interest rate is realized, r_t, the central bank and the treasury choose their policies. In our setup neither the central bank, nor the treasury have access to a commitment technology, and it is standard that we look for a Markov-perfect equilibrium – see, for instance, Nakata (2018). Therefore, I solve the problem with backward induction, starting from the sequential problem of the treasury, and then turn to the central bank's sequential problem in period t.

First, having observed the shock and the choices of the central bank and the treasury, the private sector forms expectations of future output gap and inflation in the last stage of periods $t \geq 0$, and chooses the remaining variables of the economy (π_t, y_t) according to the optimality conditions (the Dynamic IS relation (Eq. (2)), and the New-Keynesian Phillips curve (Eq. (1)).

After observing the shock and the central bank's monetary policy instrument, the treasury's problem in period t is

$$\max_{\{y_t, \tau_t, b_t, \ell_t\}_t^\infty} -\frac{1}{2} E_t \left\{ \sum_{k=0}^{\infty} \beta^k \left[\alpha_y^{Tr} y_{t+k}^2 + \alpha_\tau^{Tr} \tau_{t+k}^2 \right] \right\} \tag{13}$$

subject to

$$
\begin{aligned}
y_t = E_t y_{t+1} - (i_t - E\pi_{t+1} - r_t) + \gamma b_t, & \qquad \forall t;\\
b_t = \ell_t + (1+r)\, b_{t-1} - \tau_t, & \qquad \forall t;\\
\ell_t \geq 0, \quad \tau_t \geq 0, \quad 0 \leq b_t \leq \bar{b} < \infty, & \qquad \forall t;\\
b_t \times \mathbf{1}_{\{r \neq \underline{r}\}} = 0, \quad \ell_t \times \mathbf{1}_{\{r \neq \underline{r}\}} = 0, \quad \tau_t \times \mathbf{1}_{\{r = \underline{r}\}} = 0, & \qquad \forall t;
\end{aligned}
$$

and for an initial b_{t-1}, shock r_t, and taking current central bank's choice i_t and future paths $\{y_{t+k}, \pi_{t+k}, i_{t+k}, \tau_{t+k}, b_{t+k}, \ell_{t+k}\}$ as given with $k \geq 1$.

In this case, problem 13 shows how the treasury determines optimal fiscal policy, where its losses are functions of output gap (with path determined by the DIS), and tax distortions (subject to the treasury's budget constraint and additional restrictions on the fiscal policy instruments). The problem depends on the other agency's choices (the central bank's nominal interest rate), and since the treasury has no access to a commitment technology it takes as given the future paths of the variables of the economy. Finally, it is important to point out that the debt limit will never bind in the equilibrium characterization.

After the period-t shock, the central bank chooses its nominal interest rate, internalizing the treasury's choice of fiscal instruments. Therefore, its problem in every period t becomes

$$
\max_{\{y_t, \pi_t, i_t\}_t^{\infty}} -\frac{1}{2} E_t \left\{ \sum_{k=0}^{\infty} \beta^k \left[\alpha_y^{CB} y_{t+k}^2 + \alpha_\pi^{CB} \pi_{t+k}^2 \right] \right\} \tag{14}
$$

subject to

$$
\begin{aligned}
\pi_t &= \kappa y_t + \beta E_t \pi_{t+1}, & \forall t;\\
y_t &= E_t y_{t+1} - (i_t - E_t \pi_{t+1} - r_t) + \gamma b_t, & \forall t;\\
& i_t \geq 0, & \forall t;
\end{aligned}
$$

and for an initial b_{t-1}, shock r_t, taking future paths $\{y_{t+k}, \pi_{t+k}, i_{t+k}, \tau_{t+k}, b_{t+k}, \ell_{t+k}\}$ as given with $k \geq 1$, and while it also internalizes the treasury's choices, $\{\tau_t, b_t, \ell_t\}$, that are solution to problem 13.

Basically, problem 14 displays a discretionary central bank that minimizes its losses by choosing a nominal interest rate, i_t, subject to the zero bound. The central bank losses are functions of inflation and output gap, which also respond to the paths described by the NKPC and the DIS that belong to the private sector – and which are constraints in the CB's problem. Moreover, problem 14 also depends on the treasury's fiscal policy choices (which the central bank internalizes as they are made after the central bank chooses monetary policy). Finally, the central bank takes as given the future paths of the variables of the economy as it has no access to a commitment technology that can affect them. With these results, we can now proceed to portray our definition of equilibrium with a central bank with no forward guidance (No-FG) – which, for simplicity, I will interchangeably call the discretionary central bank equilibrium.

Definition 2 (Equilibrium with no forward guidance (No-FG)) *For every period t, an equilibrium without forward guidance consists of a set policy functions, $\{i_t, \tau_t, \ell_t, b_t\}_{t=0}^{\infty}$, and private sector allocations, $\{y_t, \pi_t\}_{t=0}^{\infty}$, such that:*

(i) The central bank solves its optimization problem (14) choosing a monetary policy instrument $\{i_t\}_{t=0}^{\infty}$,

(ii) The treasury solves its optimization problem (13) choosing fiscal policy instruments $\{\tau_t, \ell_t, b_t\}_{t=0}^{\infty}$, and

(iii) Sequences $\{y_t, \pi_t\}_{t=0}^{\infty}$ constitute a PSCE.

3.2 Equilibrium characterization

In this section I present the solutions to the problems outlined in Section 3.2.2. The solution method builds on a standard guess-and-verify mechanism (see similar applications in Christiano et al., 2011), where the equilibrium variables of the economy are supposed to have a value associated to each of the three states. Specifically, I shall argue that every variable x_t in this economy adopts three different values, $\{x_{Z,} x_R, x_S\}$. To pin down these values, I cast every optimization problem in terms of each state, and linking the resulting optimality conditions of each problem yields a system of equations whose solution verifies that the initial guess is correct.

Throughout the equilibrium characterization, I assume that the first period is $t = 0$, and the initial state of the world is $s_0 = Z$. After this period, the shock vanishes, and the recovery state follows with probability 1. If the ZLB episode does not occur again after the recovery state, the economy returns to the absorbing steady state (state S) with probability $1 - p$. With complementary probability, p, the 'ZLB-Recovery' tuple repeats for any pair of periods $t = \{k, k + 1\}$, with k even – Appendix 5 describes details of the calculations in this section.

3.2.1 Fiscal and monetary policies in each state

Treasury – State R. I first characterize the treasury's solution. In every state, the treasury solves a one-period minimization problem choosing fiscal instruments and the amount of debt to repay in the next period, taking as given the central bank's monetary policy. Throughout my exposition, I assume that the treasury repays its debt one period after debt issuance.

In the Recovery times state the treasury is assumed to: (i) Set zero lump-sum transfers, and (ii) retire past debt with taxation so as to carry no debt to the next period. Therefore, during Recovery times, τ_t, is not controlled by the treasury (taxes only absorb past debt); i.e., b_t and ℓ_t are fixed at 0. Specifically, in state R, the treasury's problem (13) can be compactly re-expressed as

$$\max_{\{b_R, \ell_R, \tau_R\}} -\frac{1}{2}\left(\alpha_y^{Tr}(y_R)^2 + \alpha_\tau^{Tr}(\tau_R)^2\right)$$

subject to

$$\begin{aligned}
&y_R = y_R^e - (i_R - \pi_R^e - r_R) + \gamma b_R \\
&b_R = (1+r)\, b_Z - \tau_R + \ell_R \\
&\ell_R = 0 \\
&b_R = 0 \\
&0 \leq b_R \leq \bar{b},\ \ell_R \geq 0,\ \tau_R \geq 0
\end{aligned}$$

where $\{i_R, b_Z, y_R^e, \pi_R^e, r_R, r_S\}$ are given. Plugging the assumptions imposed on ℓ_R and b_R, fiscal policy is determined by

$$b_R = 0, \tag{15}$$

$$\ell_R = 0 \tag{16}$$

and

$$\tau_R = (1+r)\, b_Z \tag{17}$$

For positive levels of debt during the liquidity trap (i.e., $b_Z > 0$), the last expression shows that the budget constraint instructs the treasury to use taxes during Recovery times to retire outstanding debt plus interest. Combining this with the fact that y_t is determined by the optimality conditions of the private sector (DIS and NKPC) together with expectations of future variables (y_t^e and π_t^e), and the central bank's choice of optimal monetary policy (i_t), the minimization problem for the treasury in state R is trivial. Thus, at the recovery state, the treasury's losses become

$$l_R^{Tr} \equiv -\frac{1}{2}\left(\alpha_y^{Tr}(y_R^e - i_R + \pi_R^e + r_R)^2 + \alpha_\tau^{Tr}((1+r_R)\, b_Z)^2\right)$$

Central bank – State R**.** In the Recovery state, the problem for the central bank is

$$\max_{\{y_R,\pi_R,i_R\}} -\frac{1}{2}\left(\alpha_y^{CB}(y_R)^2+\alpha_\pi^{CB}(\pi_R)^2\right), \tag{18}$$

subject to

$$\begin{aligned} \pi_R = & \quad \kappa y_R + \beta \pi_R^e, \\ y_R = & \quad y_R^e - (i_R - \pi_R^e - r_R) + \gamma b_R, \\ i_R \geq & \quad 0, \end{aligned}$$

and also subject to the treasury's choice of debt that the central bank internalizes, given by Eq. (15). In addition, the central bank takes as given private sector expectations (which I denote rational expectations with $\pi_\zeta^e \equiv E_t\{\pi_{t+1}|\, s_t=\zeta\}$ and $y_t^e \equiv E_t\{y_{t+1}|\, s_t=\zeta\}$, where $E_t\{x_{t+1}|\, s_t=\zeta\}$ is the expectation of variable x_{t+1}.). Calling λ_R^3 the multiplier of the inequality constraint (the zero lower-bound on i_R), F.O.C.s yield the well-known 'output-gap targeting' rule,

$$y_R = -\frac{\alpha_\pi^{CB}\kappa}{\alpha_y^{CB}}\pi_R - \frac{1}{\alpha_y^{CB}}\lambda_R^3. \tag{19}$$

with the complementary slackness condition $\lambda_R^3 i_R = 0$ – the solution is relegated to the Appendix, Lemma 13, state R. Recalling that the central bank prefers both low inflation and output gap, Eq. (19) indicates that the central bank prevents large losses by stabilizing any changes in inflation with opposite changes in the output gap. Abstracting momentarily from the role of multiplier λ_R^3, Eq. (19) tells that the larger the importance of output gap relative to the importance of price stability (i.e., an increasing α_y^{CB} relative to α_π^{CB}), then the central bank prefers a lower output gap distortion y_R relative to inflation π_R. The nature of this response comes from the preference structure of the monetary authority: The quadratic nature of the central bank's preferences exponentially penalizes the distortions in each variable in response to the shock. It is in the interest of the central bank, then, to exploit the given economy's environment (the NKPC) to transfer a portion of the increase in the output gap to current inflation, so as to balance the dispersion by allocating it in two variables, π_t and y_t, rather than in the output gap alone. This will eventually decrease the magnitude of the dispersion of both variables and, hence, in central bank's losses. Moreover, the central bank will be able to further absorb that shock with a variable whose value is not relevant for the monetary authority, the nominal interest rate, as long as it does not hit the zero floor: When that is the case, the equivalence in Eq. (19) breaks down, as we shall see in the analysis of the economy in state Z.

The previous relation between output gap and inflation can be maintained with strict equality as long as the ZLB constraint is not binding ($\lambda_R^3 = 0$), which is the case when the economy is at the Recovery state. Away from a liquidity trap, the optimality condition becomes

$$y_R = -\frac{\alpha_\pi^{CB}\kappa}{\alpha_y^{CB}}\pi_R. \tag{20}$$

The non-binding ZLB constraint means that at the prevailing r_R the central bank can exactly offset changes in inflation with changes in output gap. The central bank achieves so by controlling the nominal interest rate (yet to be determined but denoted i_R now), which absorbs the current r_R to keep central bank losses low as we referred above. Specifically, using the NKPC inflation and output gap levels become

$$\pi_R = \tfrac{\alpha_y^{CB}}{\alpha_y^{CB}+\alpha_\pi^{CB}\kappa^2}\beta^{PS}\pi_R^e, \qquad y_R = -\tfrac{\alpha_\pi^{CB}\kappa}{\alpha_y^{CB}+\alpha_\pi^{CB}\kappa^2}\beta^{PS}\pi_R^e. \tag{21}$$

Plugging the optimality condition (21) in the DIS constraint yields the nominal interest rate in the Recovery state implied by the targeting rule,

$$i_R = \underbrace{\gamma \overbrace{b_R}^{=0}}_{\text{Fiscal policy}} + \underbrace{y_R^e + \left(\frac{\alpha_y^{CB} + \alpha_\pi^{CB}\kappa^2 + \alpha_\pi^{CB}\kappa\beta}{\alpha_y^{CB} + \alpha_\pi^{CB}\kappa^2}\right)\pi_R^e}_{\text{Expectations effect}} + \underbrace{r_R}_{\text{Shock}} \quad \text{and} \quad i_R \geq 0. \tag{22}$$

The previous expression holds for given π_R^e, y_R^e, and b_R (variable b_R denotes the treasury's choice of debt in state R to be paid in the next period), and where the latter inequality indicates that this nominal interest rate is not constrained by the ZLB. Finally, and in order to proceed with interpretations below, observe that Eq. (22) explicitly shows b_R (although b_R is zero in state R).

The central bank's choice given by Eq. (22) shows three channels affecting the central bank's choice of interest rates, i_R. The first channel shows that that a higher natural interest rate rises nominal interest rates. For instance, a shock that drives up PS's willingness expand consumption increases aggregate demand, and will therefore put upward pressure on prices. To preserve price stability, the central bank rises nominal interest rates and, thus, it absorbs that response in aggregate demand.

Second, fiscal policy can also affect positively the optimal choice of the nominal interest rate. The intuition is that if the treasury conducts an expansionary fiscal policy, then debt-financed lump-sum transfers will benefit living cohorts via a wealth effect captured with term γb_R, which increases consumption – and, therefore, y_R.[22] But, at the same, if that level of output gap were to exceed the level the central bank prefers, then the central bank rises i_R to moderate the output gap distortion.

[22] I refer to output gap and consumption interchangeably since market clearing holds in the DIS and NKPC.

Finally, the third channel shows that the forward looking private sector's expectations over output gap and inflation can pose inflationary pressures today. When prices are expected to rise in the future, then the PS seizes a relatively larger purchasing power today that triggers current consumption today. But a larger consumption today translates into higher prices, and therefore the central bank rises the nominal interest rate.

Treasury – State Z. At the ZLB state, the treasury takes as given the central bank choices and private sector expectations. In this state, however, the treasury finances lump-sum transfers with debt issuance instead of taxation – i.e., $\tau_Z = 0$. For any $k \geq 1$, call $T = 2k$ the random date at which the economy reverts back to the absorbing state S. Then, for all even periods t with $0 \leq t < T$, I derive optimal fiscal policy for the liquidity trap state implementing the assumptions from states Z and R, and adjusting the constraints accordingly in order to cast the treasury's problem in shorter form as a function of debt b_Z. Therefore, the treasury's problem (13) can be compactly re-expressed as

$$\max_{\{b_Z,\ell_Z,\tau_Z\}} -\frac{1}{2}\alpha_y^{Tr}\left(y_Z^e - i_Z + \pi_Z^e + r_Z + \gamma b_Z\right)^2 - \beta^{Tr}\frac{1}{2}\left(\alpha_y^{Tr}\left(y_R^e - i_R + \pi_R^e + r_R\right)^2 + \alpha_\tau^{Tr}\left(\left(1 + r_R\right) b_Z\right)^2\right) \tag{23}$$

subject to

$$\begin{aligned} &0 \leq b_Z \leq \bar{b}, \\ &l_{t+1}^{Tr} \equiv l_{t+1}^{Tr}\left(y_{t+1}, \tau_{t+1}\right) \equiv l_{t+1}^{Tr}\left(i_{t+1}, b_t, b_{t+1}, \ell_{t+1}, y_{t+1}^e, \pi_{t+1}^e, r_{t+1}\right) \end{aligned}$$

and given $\{r_Z, i_Z, y_Z^e, \pi_Z^e\}$ (for the first term of the objective) and $\{i_R, b_R, \ell_R, \tau_R, y_R^e, \pi_R^e, r_R\}$ (for the second term of the objective). It is worth pointing out that the choice of b_Z could only affect periods t and $t+1$, and that (i) debt is assumed to be honoured in every period, but also that (ii) the treasury is discretionary (additional terms in the objective function belonging to periods $t+2, t+3, \ldots$ are omitted).

F.O.C. w.r.t. b_Z yields the treasury's optimal choice of debt at the ZLB state,

$$b_Z = \Phi\,(\underbrace{i_Z}_{\text{Monet. pol.}} - \overbrace{(y_Z^e + \pi_Z^e)}^{\text{PS's Expect.}} - \underbrace{r_Z}_{\text{Shock}}), \tag{24}$$

with $\Phi \equiv \frac{\alpha_y^{Tr}\gamma}{\alpha_y^{Tr}\gamma^2 + \beta^{Tr}\alpha_\tau^{Tr}(1+r_R)^2}$, and where taxes are set at zero by assumption,

$$\tau_Z = 0. \tag{25}$$

Finally, from the treasury's budget constraint we can recover the lump sum transfers, which have functional form exactly equal to b_Z,

$$\ell_Z = \Phi \left(i_Z - \left(y^e_Z + \pi^e_Z \right) - r_Z \right). \tag{26}$$

The optimality condition (24) for the treasury depends on the interaction of three different terms.

Abstracting from the zero floor on nominal interest rates, note first that the interpretation of the last negative term is straightforward: The more the PS wants to save today, the more expansionary fiscal policies will be.

Second, the positive sign on i_Z (although i_Z will be zero in equilibrium) implies that a more expansionary monetary policy during a recession (a lower i_Z) triggers a less expansionary fiscal policy from the treasury. Intuitively, since a monetary expansion increases both inflation and output gap, the treasury needs to exert less fiscal effort after a shock to mitigate output drops that affect its preferences. The monetary response, however, will have a shock portrayed by the ZLB.

Finally, observe that PS' expectations affect negatively debt choices. Since DIS and NKPC are forward looking, when the private sector expects inflation and/or output gap to be increasing in the next period, then output today (which is the variable that matters for the treasury in its losses) deviates from its target in the same direction. In response to this, expression (24) instructs the treasury to counter expectations of inflation and output growth with contractionary fiscal policy in order to close the output gap today – and mitigate losses.

Further interpretations can be drawn from analyzing the parameters involved in the coefficient of the treasury's optimal choice of debt. Abstracting from PS's expectations and CB's policy choice, without loss of generality we can first multiply both sides of Eq. (24) by γ to obtain

$$\gamma b_Z = \gamma \Phi \left(i_Z - y^e_Z - \pi^e_Z - r_Z \right),$$

where

$$\gamma \Phi \equiv \frac{\alpha^{Tr}_y \gamma^2}{\alpha^{Tr}_y \gamma^2 + \beta^{Tr} \alpha^{Tr}_\tau \left(1 + r_R\right)^2}$$

Since the LHS is the same term that enters in the state-Z Dynamic IS, the above expression allows us to motivate an interpretation in terms of output gap – note that γb_Z measures the impact of fiscal policy translated in terms of y_Z. For any negative r_Z, the numerator of the above expression shows that the response of the LHS, γb_Z, (measured in y_Z terms as per the DIS) is increasing both on (i) the treasury's weight on output, α^{Tr}_y, and (ii) the magnitude of the wealth effect generated by debt financed fiscal policy, γ^2 – every unit of additional debt b_Z maps into y_Z in increments of γ, and it is adjusted to the power of 2 since preferences are quadratic. However, the expansionary impact of fiscal policy generate a cost. In particular, this cost is portrayed by the denominator, where term $\alpha_y^{Tr}\gamma^2 + \beta^{Tr}\alpha_\tau^{Tr}(1+r_R)^2$ disciplines the magnitude of b_Z since fiscal policy will have to be repaid in Recovery times. Specifically, the second additive term in the denominator highlights that tomorrow's (discounted) debt yields a quadratic cost $(1+r_R)^2$, which will introduce further distortions in the treasury's loses tomorrow. Hence, debt-financed fiscal policy is less desirable the more averse to tax collection costs the treasury is: If α_y^{Tr} is extremely high, then negative r_Z yields close to zero response of debt financed fiscal policy (and, therefore, the treasury does not mitigate a recession). On the contrary, if the treasury cares little about tax collection costs in the economy (α_τ^{Tr} tends to 0), then the above expression simply becomes $\gamma b_Z = i_Z - y_Z^e - \pi_Z^e - r_Z$ and the treasury absorbs a larger proportion of the shock. In summary, Eq. (24) indicates that the choice of debt in one period inevitably affects relevant treasury's outcomes in the next period, and because of that the treasury absorbs the shock by balancing the amount of distortion that it generates between these two periods.

Central bank – State Z. When there is a large negative shock to the natural interest rate, the problem for the central bank in state Z is similar to problem (18). The differences, however, are that (i) now we are looking at the case where the ZLB constraint binds; $\lambda_Z^3 > 0$, and (ii) the treasury's choice of debt is different from zero, with expression given by the optimality condition; (24).

By the complementary slackness condition, a positive Lagrange multiplier implies that

$$i_Z = 0 \tag{27}$$

during the liquidity trap state, and the F.O.C. becomes

$$(1-\gamma\Phi)\left(\alpha_y^{CB} y_Z + \alpha_\pi^{CB}\kappa\pi_Z\right) < 0. \tag{28}$$

Eq. (28) states that the targeting rule of the Central bank fails to both absorb and balance the fluctuations in inflation and the output gap when a shock to the natural interest rate drives the nominal interest rate to the ZLB. Linking the ZLB constraint in the DIS and the NKPC renders the associated inflation and output gap values in state Z,

$$\pi_Z = ((1-\gamma\Phi)\kappa) y_Z^e + \left((1-\gamma\Phi)\kappa + \beta^{PS}\right)\pi_Z^e + \kappa(1-\gamma\Phi) r_Z,$$

$$y_Z = (1-\gamma\Phi) y_Z^e + (1-\gamma\Phi)\pi_Z^e + (1-\gamma\Phi) r_Z.$$

The expressions for π_Z and y_Z described above are in stark contrast with our previous results for π_R and y_R in (21): While those were only a function of PS expectations, the above expressions for π_Z and y_Z indicate that inflation and the output gap in state Z are also affected by the negative discount factor shock, r_Z, as well as parameters from fiscal policy embodied in Φ.

Finally, and in order for the ZLB to bind in equilibrium, the shock needs to satisfy the following technical condition (which is assumed to hold from now on).

Condition 3 (C1) *The nominal interest rate in state Z, i_Z, is zero if*

$$r_Z < -y_Z^e - \left(1 + \frac{\alpha_\pi^{CB}\kappa\beta^{PS}}{\alpha_y^{CB} + \alpha_\pi^{CB}\kappa^2}\frac{1}{1-\gamma\Phi}\right)\pi_Z^e. \tag{29}$$

Condition C1 characterizes which values of r_Z make the ZLB bind. It is worth to point out that moving all terms to the left-hand-side of the inequality yields the exact same expression of the optimal nominal interest rate choice as in Eq. (22), with the only difference that it is evaluated at the Z state instead of R,

$$y_Z^e + \left(\frac{\alpha_y^{CB} + \alpha_\pi^{CB}\kappa^2 + \alpha_\pi^{CB}\kappa\beta}{\alpha_y^{CB} + \alpha_\pi^{CB}\kappa^2}\right)\pi_Z^e + r_Z + \gamma\overbrace{\Phi\left(-y_Z^e - \pi_Z^e - r_Z\right)}^{=b_Z} \equiv \widetilde{i}_Z \tag{30}$$

Expression (30) represents an *implicit* optimal nominal interest rate. It is implicit as it defines what the CB's nominal interest rate choice would be in state Z, absent the ZLB constraint. In equilibrium, however, $\widetilde{i}_Z$ will not be allowed to happen: For given y_Z^e and π_Z^e, the shock, r_Z, will be negative enough to satisfy expression (29) which, in turn, means that $i_Z = 0$ prevails. To complete the requirement prescribed in Eq. (29), we need to calculate the equilibrium outcomes y^e, and π_Z^e – in my numerical exercise I verify that this condition holds in equilibrium.

Treasury – State S**.** In state S the minimization problem of the treasury is trivial. In particular, during state S the treasury still pays past debt with taxation, and carries no debt to the next period. Therefore, at any period t with state S that follows after an S- or R-state, fiscal policy is given by

$$b_S = 0, \tag{31}$$

$$\ell_S = 0, \tag{32}$$

$$\tau_S = 0. \tag{33}$$

Central bank – State S. When the economy is at state S, it reaches an absorbing state (and there is no further uncertainty onward). Hence, at state S, the solution to the central bank's general problem (14) is similar to the solution in state R; i.e.,

$$\pi_S = \tfrac{\alpha_y^{CB}}{\alpha_y^{CB}+\alpha_\pi^{CB}\kappa^2}\beta^{PS}\pi_S^e, \qquad y_S = -\tfrac{\alpha_\pi^{CB}\kappa}{\alpha_y^{CB}+\alpha_\pi^{CB}\kappa^2}\beta^{PS}\pi_S^e, \tag{34}$$

and

$$i_S = y_S^e + \left(\tfrac{\alpha_y^{CB}+\alpha_\pi^{CB}\kappa^2+\alpha_\pi^{CB}\kappa\beta}{\alpha_y^{CB}+\alpha_\pi^{CB}\kappa^2}\right)\pi_S^e + r_S \quad \text{and} \quad i_S \geq 0. \tag{35}$$

Private sector – States $\{Z, R, S\}$. In each state, the private sector forms rational expectations that close the determination of the economy's variables (π_t, y_t). Specifically, given the Markov structure for the states of the world, when the PS makes its decisions in state R it forms rational expectations,

$$y_R^e = py_Z + (1-p)\,y_S \text{ and } \pi_R^e = p\pi_Z + (1-p)\,\pi_S. \tag{36}$$

Eqs. (36) use that the economy transitions from state R to Z with probability p. (After analyzing state S, each equality will later feature the fact that the value of the output gap and inflation rate in state S are zero.) Likewise, in states R and S the rational expectations of the private sector are such that

$$y_Z^e = y_R \text{ and } \pi_Z^e = \pi_R, \tag{37}$$

and

$$y_S^e = y_S \text{ and } \pi_S^e = \pi_S. \tag{38}$$

3.2.2 Equilibrium without forward guidance

We are now ready to characterize the equilibrium of the economy when the central bank is discretionary and the treasury chooses debt-financed fiscal policy during liquidity traps to mitigate recessions and deflations. Using a backward induction logic, I start characterizing the economy's equilibrium at the steady state, and then implement this result into the equilibrium variables from states R and Z. In what follows, recall that I am attempting a solution to a discrete stochastic problem using the linearized version of the equations of the economy (see, for example, Eggertsson, 2011b). I use stars to denote equilibrium outcomes without forward guidance.

Consider T a random period of time where the shock to the natural interest rate returns to its steady state, $r_t = r_S$. Since at any period $t \geq T$ there is no uncertainty and the problem becomes deterministic, the economy achieves the steady state equilibrium. In line with E11Z, our assumptions on fiscal and monetary policies imply that fiscal and monetary instruments are perfectly correlated with the shock. Hence, in every $t \geq T$ fiscal policy is set at $(\tau_t, \ell_t, b_t) = (0, 0, 0)$.

It is worth pointing out that setups with Taylor rules augmented with a ZLB on nominal interest rates can lead to the existence of multiple equilibria – see, for example, Benhabib et al. (2001). These setups depict polar cases characterized by the zero-inflation and zero-output gap, and by a self-fulfilling deflationary equilibrium as the other one, namely, a perpetual liquidity trap. Moreover, this result is also present in a discretionary-central-bank setup like ours – See Appendix 5. To focus our analysis of fiscal and monetary interactions only during liquidity traps, I assume that the central bank has the ability to guarantee the zero-inflation steady state – for a similar assumption, see Werning (2012); for a selection mechanism to rule out the self-fulfilling deflationary equilibrium, see Jung et al. (2005). Therefore, when the economy reaches the absorbing steady state, the natural interest rate $r_t = r_S > 0$, and the system of equations formed with the monetary and fiscal optimal policies in the S state (resp. Eqs. (35), (31), (33) and (32)), together with the economy's variables y_t and π_t determined in S by the private sector's optimality conditions DIS and NKPC and expectations (Eq. (38)) yields a steady state characterized by equilibrium outcomes

$$\pi_S^* = 0,\ y_S^* = 0,\ i_S^* = r_S,\ \tau_S^* = 0,\ \ell_S^* = 0,\ b_S^* = 0,\ \lambda_S^{1,*} = 0,\ \lambda_S^{2,*} = 0,\ \lambda_S^{3,*} = 0 \tag{39}$$

(where the PS rationally expects $(\pi_t^e, y_t^e) = (\pi_S, y_S) = (0, 0)$) – see Claim 15 in Appendix. It is worth pointing out that the solution highlights that the central bank achieves zero inflation and zero output gap by fully absorbing the shocks to the economy with the nominal interest rate.

Finally, given our model assumptions together with Condition (29), the system of equations formed with the steady state outcomes (39) and the monetary and fiscal optimal policies in the ZLB state and in the Recovery state (resp. Eqs. (27), (24), (25), (26) and Eqs. (22), (15), (17) (16)), together with the economy's variables y_t and π_t determined by the private sector's optimality conditions DIS and NKPC in the ZLB state and in the Recovery state, and private sector expectations in the ZLB state and in the Recovery state (resp. Eq. (37) and Eq. (36)) render the equilibrium outcomes of the model economy when the central bank cannot implement credible forward guidance.

Proposition 4 *The solution to the system formed with the equations characterizing the economy when the central bank cannot implement credible forward guidance during a liquidity trap is defined by the following equilibrium outcomes: In states* Z, R, *and* S,

$$\pi_Z^* = \theta_Z^\pi r_Z, \qquad y_Z^* = \theta_Z^y r_Z, \qquad i_Z^* = 0, \qquad b_Z^* = \theta_Z^b r_Z, \qquad \ell_Z^* = \theta_Z^\ell r_Z, \qquad \tau_Z^* = 0,$$

$$\pi_R^* = \theta_R^\pi r_Z, \qquad y_R^* = \theta_R^y r_Z, \qquad i_R^* = \theta_R^i r_Z + r_R, \qquad b_R^* = 0, \qquad \ell_R^* = 0, \qquad \tau_R^* = \theta_R^\tau r_Z,$$

$$\pi_S^* = 0, \qquad y_S^* = 0, \qquad i_S^* = r_S, \qquad b_S^* = 0, \qquad \ell_S^* = 0, \qquad \tau_S^* = 0,$$

with state-Z coefficients $\theta_Z^\pi \equiv \left[\alpha_y^{CB} + \alpha_\pi^{CB}\kappa^2\right]\kappa\upsilon/\phi$, $\theta_Z^y \equiv [\alpha_y^{CB}(1 - \left(\beta^{PS}\right)^2 p) + \alpha_\pi^{CB}\kappa^2]\upsilon/\phi$, $\theta_Z^b \equiv -\alpha_y^{Tr}\gamma(\alpha_y^{CB}(1 - p\left(\beta^{PS}\right)^2) + \kappa^2\alpha_\pi^{CB})/\phi$, *and* $\theta_Z^\ell = \theta_Z^b$, *and state-R coefficients* $\theta_R^\pi \equiv \left[\alpha_y^{CB}\right]\kappa\beta^{PS}p\upsilon/\phi$, $\theta_R^y \equiv -\left[\alpha_\pi^{CB}\kappa^2\right]\beta^{PS}p\upsilon/\phi$, $\theta_R^i \equiv (\alpha_y^{CB}((1 - \left(\beta^{PS}\right)^2 p) + \kappa) + \alpha_\pi^{CB}\kappa^2(1 + \kappa + \beta^{PS}))p\upsilon/\phi$, *and*

$$\theta_R^\tau \equiv \frac{\left(-\alpha_y^{CB}\left(\left(1-p\left(\beta^{PS}\right)^2\right)\gamma^2\alpha_y^{Tr}+\left(1-p\beta^{PS}\left(\beta^{PS}+\kappa\right)\right)\upsilon\right)-\alpha_\pi^{CB}\kappa^2\left(\gamma^2\alpha_y^{Tr}+\upsilon\right)\right)(1+r_R)\gamma\alpha_y^{Tr}}{\phi\left(\alpha_y^{Tr}\gamma^2+\upsilon\right)},$$

where $\upsilon \equiv \alpha_\tau^{Tr}\beta^{Tr}(1 + r_R)^2$ *and* $\phi \equiv \alpha_y^{CB}((1 - p\left(\beta^{PS}\right)^2)\gamma^2\alpha_y^{Tr} + (1 - p\beta^{PS}(\beta^{PS} + \kappa))\upsilon) + \alpha_\pi^{CB}\kappa^2(\gamma^2\alpha_y^{Tr} + (1 + \beta^{PS}p)\upsilon)$.

Proof. See proof in Appendix 5. ∎

The previous result characterizes the equilibrium outcomes of the economy in closed form. The Proposition also proves our initial guess from our solution method that there exist solutions for the values of the variables at different states. The following remarks from Proposition 4 are in order. First, observe that the numerator of variables y_R and π_R depend on the probability of reversion to the ZLB, p, while variables y_Z and π_Z do not. In particular, observe that y_R and π_R drops to zero as this probability approaches zero from above. This result is compatible with the literature (e.g., see Eggertsson (2011b)), and it implies that, when $p \to 0$, then the economy at state R replicates the steady state outcomes. Intuitively, this means that households and firms in the Private Sector expect the shock to vanish, which makes them expect the central bank to be always able to absorb shocks with the nominal interest rate – and, therefore, zero inflation and output gap are realized.

Second, observe also that, apart from their difference in p, the numerators of inflation and output gap, π_R and y_R, have: (1) A common factor in the numerator, $\alpha_\tau^{Tr}\beta^{Tr}\left(1 + r_R\right)^2$; and (2) a discrepancy captured by the parameters in square brackets. These two factors reveal that inflation and output gap outcomes at each period t feature a combination of a 'fiscal policy content' (term $\alpha_\tau^{Tr}\beta^{Tr}\left(1 + r_R\right)^2$) and a 'monetary policy content' (the non-common factor in square brackets), where the latter differs with the state. When monetary policy is away from the ZLB, the equilibrium nominal interest rate, i_R^*, is chosen by the central bank to absorb the current natural interest rate shock. This way, the central bank can optimally balance the contemporary feedback channel that exists between inflation and output gap at the current period (given by the slope, κ, of the NKPC), where the only feedback channel between inflation and output gap that the CB cannot affect is the expectations channel (due to the discretionary nature of the monetary authority). On the contrary, at the ZLB, the equilibrium nominal interest rate, i_Z^*, falls short of absorbing the exogenous shock, r_Z, and therefore the monetary policy content of inflation and output gap is a function of both inflation and output gap parameters simultaneously. This is because the CB exhausts the nominal interest rate when it reaches the zero floor in its attempt to prevent the contemporary feedback channel from the Phillips curve: The nominal interest rate becomes zero, and the unabsorbed portion of the shock to r_t cascades back to y_Z^*, and therefore the CB cannot even prevent the contemporaneous feedback existing between y_t and π_t via the NKPC that adds up an additional distortion to its losses – and which also adds up to the already-existing distortion in inflation and output gap due to the expectations channel.

Finally, observe that the wealth effect from debt-financed fiscal policy that enters in the DIS, γ, directly affects the choice of debt from the treasury, b_Z^*, as well as the equilibrium outcomes for inflation and the output gap. In particular, observe that, if $\gamma = 0$, then $b_Z^* = 0$. The intuition is that, if there is no wealth effect to the PS coming from fiscal policy, then it is not in the treasury's interest to worsen its losses (via taxes) by issuing lump-sum transfers backed with debt during a liquidity trap. This conclusion is compatible with Ricardian economies, and it would be equivalent to having an economy where only the central bank can stabilize macro variables. But, by the same token, as γ approaches zero, output gap and inflation also stop being sensitive to fiscal policy. Specifically, as $\gamma \to 0$, then $\phi \to \left[\alpha_y^{CB}\left(1 - p\beta^{PS}\left(\beta^{PS} + \kappa\right)\right) + \alpha_\pi^{CB}\kappa^2\left(1 + \beta^{PS}p\right)\right]\left\{\alpha_\tau^{Tr}\beta^{Tr}\left(1 + r_R\right)^2\right\}$, and the factor in braces cancels out with the same factor present in each numerator of the equilibrium outcomes – the fiscal policy content of $(y_Z^*, \pi_Z^*, y_R^*, \pi_R^*)$. As a result of this, variables in Proposition 4 are independent of any fiscal parameter, and our equilibrium characterization matches with that in the literature where a discretionary central bank faces deflation both at the ZLB (Eggertsson and Woodford (2003)), and away from the ZLB (Nakov (2008); Nakata (2018); Walsh (2018)).

When $\gamma > 0$, instead, the impact of fiscal policy will be relevant to the economy's variables. The next section introduces a graphical analysis of these results that will help identify the response of the equilibrium outcomes to a large negative shock when the central bank cannot implement credible forward guidance.

3.3 Numerical exercise

Assuming that the central bank can only use the nominal interest rate and has no access to forward guidance, I now proceed to show paths for the equilibrium outcomes in periods 0 and 1 – which represent the equilibrium outcomes for any pair of consecutive periods with states (Z, R). I plot figures associated to a numerical example with quarterly parameters set to standard values in the literature – it is worth pointing out that the following parameterization does not represent a calibration as some of the economy's parameters are freely chosen.

Weights α_π^{CB} and α_τ^{Tr} are set to 1.5 and 0.5 relative to an output gap weight normalized at 1. This choice aims to capture a conservative monetary authority (i.e., a central bank relatively more concerned with price stability), and a fiscal authority less concerned about tax distortions relative to output gap distortions. I set the discount factor β to 0.99, which is equivalent to a steady state natural interest rate level (annualized) of approximately 4 percent. In order to generate a recession and deflation at the ZLB and along the lines of Christiano et al. (2011), I assume a shock that increases households' discount factor and, therefore, lowers the natural interest rate, $\underline{r}$, to -0.02 (annualized). The shock lasts for one period, and after that period the natural interest rate reverts back to $r \equiv -\ln\beta$ in states $\{R, S\}$. In the model I also set the coefficient of government debt in the dynamic IS curve, γ, at values 0 (inactive fiscal policy) and 0.5 (active fiscal policy case),[23] and κ to approximately 0.033. Finally, in state R, the ZLB occurs again with probability p set at 0.2.

Parameter	Value	Description	Source
β	0.99	Discount factor (quarterly; $r \approx 0.01$)	D10, CD10, C11
r_Z	-0.005	Natural interest rate shock (quarterly)	EW03, C11, MNS17, ST18
γ	$\{0, 0.5\}$	Dynamic IS coefficient of b_t	D10
κ	0.0329	Phillips Curve coefficient of y_t	D10, C11
α_π^{CB}	1.5	Central Bank's weight on inflation	(Free)
α_y^{CB}	1	CB's weight on output gap	(Free)
α_τ^{Tr}	0.5	Treasury's weight on tax distortions	(Free)
α_y^{Tr}	1	Treasury's weight on output gap	(Free)
p	0.2	Probability of returning to ZLB from R	(Free)

Table 1: Parameters. Sources: EW03: Eggertsson and Woodford (2003); D10: Devereux (2010); CD10: Cook and Devereux (2010); MNS17: McKay et al. (2017); C11: Christiano et al. (2011); ST18: Smets and Trabandt (2018). Equal values assigned for $\{r_S, r_R, r\}$.

Starting from the zero inflation and zero output gap Steady State, Figure 3 shows the equilibrium outcomes for a sudden negative shock to the natural interest rate at time 0, which

[23] The parameterization of γ was chosen for convenience in order to magnify the responses. The effects are qualitatively the same when carried out using parameterizations from the literature (e.g., 0.011 from Devereux (2010); and 0.01 from Kirsanova et al. (2005)).

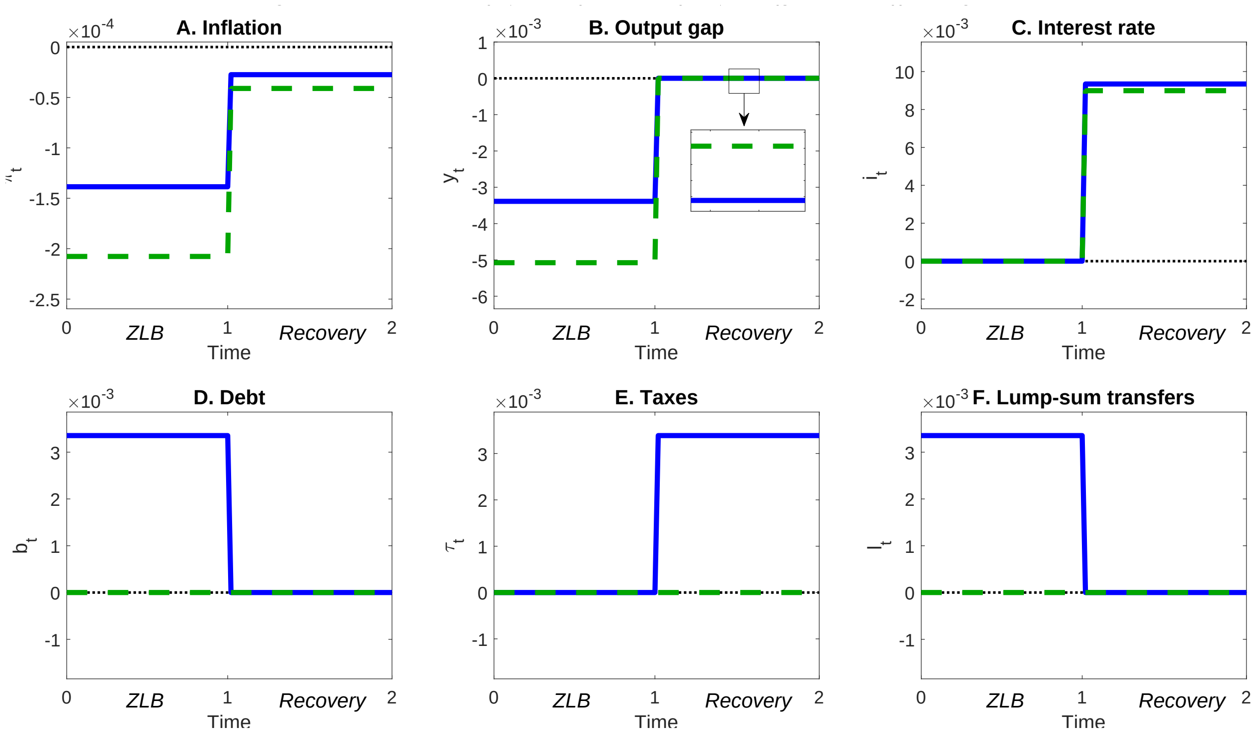


Figure 3: Simulated time series under central bank discretion. The lines represent the responses of equilibrium outcomes to a one-period shock to the natural interest rate with inactive fiscal policy, $\gamma = 0$, (dashed green lines) and active fiscal policy, $\gamma > 0$ (solid blue lines). Variables in log deviations (except i_t).

reverts back to r in period 1 (Recovery state). As portrayed in period 0 from Panel C and Panel D, the negative demand shock triggers an immediate expansionary response from fiscal and monetary authorities. Note, however, that Condition (29) holds, and so the negative demand shock drives the economy to a liquidity trap. Therefore, in period 0 we observe that the Central bank sets zero nominal interest rates (Panel C), and the treasury finds it optimal to implement debt-financed lump-sum transfers to mitigate the fall in output (Panels D and F). Note, however, that the counterpart of expansionary fiscal policy is a future rise in taxes (Panel E), and when the treasury suffers a positive cost α_{τ}^{Tr} from distortionary fiscal policy, the next-period rise in taxes triggers losses to the treasury. This means that, in equilibrium, the treasury does not counter the entire decline in output with expansionary fiscal policy, and this translates into deflation and recession at the time of the shock (Panels A and B).

The simulations in Fig. 3 show that, under inactive fiscal policy ($\gamma = 0$), the drop in inflation and output gap are worse relative to their active fiscal policy counterpart ($\gamma > 0$). In state Z, and after a negative shock to r_t, the blue solid and the green dashed lines in Figure 3 show that the existence of the ZLB constraint contributes to deflation during a liquidity trap.[24] In fact, the deflationary outcome at the ZLB is explained by the optimal CB response:

[24] This phenomenom so-called deflation bias of optimal monetary policy has been formalized in the litera-

When a negative shock to the natural interest rate forces the central bank's corner solution (i.e., $i_Z^* = 0$), variable π_Z^* inevitably absorbs part of the shock – and deflation emerges in equilibrium. This result aligns with the literature that observes deflation not as the result of a monetary policy mistake, but rather the result of a central bank that optimizes subject to credibility constraints (see Eggertsson, 2006). Analytically, result $i_Z^* = 0$ is because our parameterization satisfies condition C1 (Expression (29)): The shock to the natural interest rate is negative enough to leave the monetary authority without ammunition to prevent the entire shock, and therefore the portion of the shock that is not absorbed creates recession and deflation.[25] Eventually, there is a fraction of this portion of the shock that the treasury would like to mitigate when its policy is effective – as we shall see when $\gamma > 0$. Note also that in state R deflation is observed again, and it is linked to the central bank's inability to commit to a certain monetary policy – also labelled by Nakov (2008) as the 'deflationary bias in expectations'. Specifically, the central bank cannot commit to inflate in other periods, and this happens due to the forward-looking behavior of the PS that anticipates that the ZLB (and, therefore, deflation) may repeat in the future. As with some probability there might be a shock tomorrow, this shock may trigger a larger desire to save in that period. But since the shock is large, the PS knows that the economy will be again in a liquidity trap with its associated deflation. This further deters the PS from spending even when the economy is away from the ZLB, which triggers deflation even at the Recovery state.

The difference between the green dashed and the blue solid lines is captured by parameter γ. When $\gamma > 0$, the blue solid lines show the impact of fiscal policy in the economy. The first takeaway is that expansionary fiscal policy introduces changes in the outcomes of both states. Specifically, in state Z, fiscal policy mitigates a portion of the deflation in state Z (blue solid line above green dashed line; i.e., less negative inflation), and it closes output gap distortions (blue solid line above green dashed line in Z, meaning less recession). From the treasury's perspective, however, the fiscal authority will not counter the shock in full extent as it has to balance its preference for output drops with its dislike for tax distortions. This is because the treasury bears the total cost (in terms of tax distortions) of any expansionary fiscal policy implemented. Finally, in state R, inflation outcomes improve again (blue solid line above green dashed line; i.e., less negative inflation), as well as output gap (blue solid line below green dashed line in R, meaning a smaller distortion relative to the targeted 0 output gap target). However, we observe that this circumstance emerges with a simultaneous rise in the nominal interest rate, compared to the case of no fiscal policy, $\gamma = 0$. The intuition is

ture (see Eggertsson (2006)), and it has been shown to also be present both during and after liquidity traps (see, for instance, Nakov (2008); Nakata and Schmidt (2019); Walsh (2018)).

[25] This result alligns with the literature in that deflation is not the result of a monetary policy mistake, but rather the result of a CB optimizing subject to constraints – see Eggertsson (2006).

that in Z states active fiscal policy mitigates the recession and deflation, which also improve outcomes in R states via expectations. But as fiscal policy introduces gains in terms of inflation and output gap in state R, it eventually releases the pressure on the nominal interest rate instrument. Eventually, this release of pressure frees the central bank from using i_R^* to operate in view of the possibility of deflation in state Z, and it thus allows the monetary authority to absorb a larger portion of the state-R natural interest rate to prevent that a larger portion of r_R cascades back to y_R^* – and, eventually, to π_R^*.

4 Forward guidance and fiscal policy

In the previous section I characterized an economy with a central bank that can only use the nominal interest rate as the monetary policy instrument during a recession. In this section I analyze the same setup but when a central bank has the ability to conduct credible forward guidance. In particular, I study whether the central bank can implement credible forward guidance in the presence of fiscal policy, and how forward guidance interacts with fiscal policy compared to the equilibrium from the previous section.

In this paper I define forward guidance as a nominal interest rate that the central bank promises in states where there is a liquidity trap to implement during Recovery times. Specifically, in the equilibrium I seek to build, I want to characterize credible nominal interest rates for Recovery times that are lower relative to the nominal interest rate without access to forward guidance. By credible nominal interest rates I mean rates that the central bank announces and that the Private Sector rationally expects to be implemented after a liquidity trap ends. Since the central bank acts under discretion, forward guidance will be made credible using an equilibrium concept that exploits the repeated structure of the fiscal-monetary interaction called a *sustainable equilibrium*. Firstly introduced by Chari and Kehoe (1990), this concept has been widely used in the literature that studies equilibria when the central bank cannot access commitment technologies but it can use reputation to implement monetary policy. This is why the equilibrium is interchangeably referred to as reputational equilibrium. In a nutshell, now past actions matter, and the objective is to allow the central bank to conduct forward guidance based on a trigger strategy that involves reverting to the no forward guidance scenario from the previous section (i.e., where the central bank lacks credibility) if the central bank ever deviates. In this model I assume the PS punishes the central bank forever in case it deviates. Hence, if the central bank makes an announcement but it then deviates to the discretionary solution (i.e., it chooses the policy prescribed by the solution to its loss minimization problem instead of implementing the forward guidance level), then the central bank loses credibility forever – and the PS no longer believes the central bank, meaning that the economy reverts to the solution without forward guidance from Proposition 4 onward.

This section is organized as follows. First, I introduce relevant concepts to characterize reputational equilibria. This part will be similar to other works in the literature (e.g., Kurozumi (2008)). Specifically, it will follow very closely the exposition from Nakata (2018), but it differs from this work in that I analyze the impact of fiscal parameters on the credibility of monetary policy. Second, I will numerically explore whether an equilibrium with credible forward guidance exists when there is fiscal policy mitigating a recession. Since the answer will be that there exists such credible forward guidance equilibrium, I shall proceed to analyze how sensitive the credibility of forward guidance is to fiscal policy parameters. Finally, I will analyze whether there are any relevant fiscal and monetary policy interactions that describe the type of response (namely, subsitutability or complementarity) of a central bank and a treasury to recurring liquidity traps events.

4.1 The policy game

In what follows, I borrow the sustainable equilibrium design portrayed in Kurozumi (2008) and Nakata (2018), and follow their exposition very closely to characterize sustainable equilibria. It is worth to point out, however, that in this paper I depart from the setup in Nakata (2018) (who model the repeated interaction between a central bank and infinitely many small private agents that are strategically negligible) in that I append a treasury to this description – see Basso (2009) for a similar case with two big policymakers.[26]

Actions and histories. I shall denote with $a_t = \left(a_t^{CB}, a_t^{TR}, a_t^{PS}\right)$ the vector of actions taken by the central bank, the treasury and the private sector, respectively, where a_t^{CB} is defined by $a_t^{CB} \equiv i_t \in \mathbb{R}_{\geq 0}$, a_t^{Tr} is equivalent to $a_t^{Tr} \equiv (b_t, \tau_t, \ell_t) \in \mathbb{R}^3_{\geq 0}$, and a_t^{PS} denotes $a_t^{PS} \equiv (y_t, \pi_t) \in \mathbb{R}^2$.[27] Since now competitive equilibrium allocations will depend on these fiscal and monetary policy actions as well as on the exogenous state variable r_t (linking states of the world $\{Z, R, S\}$ to values $\underline{r} < 0$ and $r > 0$), I also need to define histories that keep track of past actions and r_t. For every $t \geq 0$ I formally define a *history* of the game, h_t, as $h_t = \left(h_{t-1}, a_{t-1}^{CB}, a_{t-1}^{Tr}, r_t\right)$ – where I assume $h_{-1} = \varnothing$, $a_{-1}^{CB} = \varnothing$, and $a_{-1}^{Tr} = b_{-1} = 0$, and $h_0 = r_0$. Given the recursive structure of h_t, it can be shown that $h_t = \left(\left(a_{-1}^{CB}, a_0^{CB}, ..., a_{t-1}^{CB}\right), \left(a_{-1}^{Tr}, a_0^{Tr}, ..., a_{t-1}^{Tr}\right), (r_0, ..., r_t)\right)$; i.e., histories are a function of (i) past monetary and fiscal actions, and (ii) past and present exogenous states. Finally, due

[26] For the sake of exposition, I also borrow the notation used in Kurozumi (2008).

[27] Private sector actions will be skipped from histories since the PS is strategically negligible – for a further discussion, see Chari and Kehoe (1990).

to the timing of decisions, we define a *current history for* $k = \{CB, Tr, PS\}$ as the vector formed with histories and contemporary events observed by k, and up to the time k makes a decision. Specifically, I shall denote a current history with h_t^k, where $h_t^{CB} \equiv h_t$, $h_t^{Tr} \equiv \left(h_t, a_t^{CB}\right)$, and $h_t^{PS} \equiv \left(h_t, a_t^{CB}, a_t^{Tr}\right)$.[28]

Strategies, continuation strategies and future histories. Be $\left\{\sigma^{CB}, \sigma^{Tr}, \sigma^{PS}\right\}$ the strategies for the central bank, the treasury and the private sector. A strategy σ^k represents a sequence of functions mapping histories for $k = \{CB\}$ (or mapping histories and contemporary actions for $k = \{Tr, PS\}$) into time-t actions. Formally, for $k = \{CB, Tr, PS\}$, a strategy for every t is defined by $\sigma^k = \left\{\sigma_t^k(\cdot)\right\}_{t=0}^{\infty}$ with element $\sigma_t^k(\cdot)$ of the sequence of functions described as follows. First, given the CB's current history, $h_t^{CB} \equiv h_t$, the Central bank uses σ^{CB} to (i) set the current nominal interest rate; $\sigma_t^{CB}\left(h_t^{CB}\right) = a_t^{CB} \equiv i_t$, and (ii) set future nominal interest rates for every future history; $\left\{\sigma_{t'}^{CB}\left(h_t^{CB}\right)\right\}_{t'>t}$ – future histories are defined later. Second, given the treasury's current history, $h_t^{Tr} \equiv \left(h_t, a_t^{CB}\right)$, the treasury uses σ^{Tr} to (i) set the current fiscal policy; $\sigma_t^{Tr}\left(h_t^{Tr}\right) = a_t^{Tr} = (b_t, \tau_t, \ell_t)$, and (ii) set future fiscal policy for every future history; $\left\{\sigma_{t'}^{Tr}\left(h_t^{Tr}\right)\right\}_{t'>t}$. Third, given the PS's current history, $h_t^{PS} \equiv \left(h_t, a_t^{CB}, a_t^{Tr}\right)$, the PS uses σ^{PS} to (i) set the current inflation and output gap; $\sigma_t^{PS}\left(h_t^{PS}\right) = \left(\sigma_t^{PS,\pi}\left(h_t^{PS}\right), \sigma_t^{PS,y}\left(h_t^{PS}\right)\right) = a_t^{PS} = (\pi_t, y_t)$, and (ii) set future inflation and output gap for every future history; $\left\{\sigma_{t'}^{PS}\left(h_t^{PS}\right)\right\}_{t'>t}$. In summary, the element $\sigma_t^k(\cdot)$ of sequences σ^k is defined as $\sigma_t^k\left(h_t^k\right) = a_t^k$. In addition, given strategies $\left\{\sigma^{CB}, \sigma^{Tr}, \sigma^{PS}\right\}$, we call σ_t^k the continuation strategy from a current history h_t^k for $k = \{CB, Tr, PS\}$. Formally, from any current history h_t^k for $k = \{CB, Tr, PS\}$, these are defined as $\left\{\sigma_t^{CB},\ \sigma_t^{Tr},\ \sigma_t^{PS}\right\} = \left\{\left\{\sigma_{t'}^{CB}\left(h_t^{CB}\right)\right\}_{t'\geq t}, \left\{\sigma_{t'}^{Tr}\left(h_t^{Tr}\right)\right\}_{t'\geq t}, \left\{\sigma_{t'}^{PS}\left(h_t^{PS}\right)\right\}_{t'\geq t}\right\}$ for every future history. Continuation strategies determine current and future monetary policy, fiscal policy and output gaps and inflation rates for every future history. Finally, I define future histories as histories generated by $\left\{\sigma^{CB}, \sigma^{Tr}\right\}$. Formally, $h_{t+1} = \left(h_t, \sigma_t^{CB}\left(h_t^{CB}\right), \sigma_t^{Tr}\left(h_t^{Tr}\right), r_{t+1}\right)$, for every $t \geq 0$.

The previous notation makes explicit fiscal, monetary and private sector actions as functions of past (or past and current) actions and past and current exogenous states. Having defined these objects, I now present the central bank, treasury and private sector problems using this notation, and then introduce the sustainable equilibrium concept. In particular, I will now show how $\left\{\sigma^{CB}, \sigma^{Tr}, \sigma^{PS}\right\}$ are determined. First, I shall start with σ^{PS} for given $\left\{\sigma^{CB}, \sigma^{Tr}\right\}$. To do this, I focus on the determination of the continuation strategy σ_t^{PS}

[28]It is important to stress that the Central Bank, the Treasury and the Private Sector observe different histories relative to each other; note that Dong and Young (2019)call this an 'expanded history'. See also Gnocchi and Lambertini (2016) for a similar exposition.

standing at any current history h_t^{PS} for every future history. Rewriting the continuation strategy σ_t^{PS} as $\sigma_t^{PS}\left(h_t^{PS}\right) \cup \sigma_{t+1}^{PS}$, then given any current history h_t^{PS}, σ_t^{PS} must be such that the current strategy at t, $\sigma_t^{PS}\left(h_t^{PS}\right) \equiv \left\{\sigma_t^{PS,\pi}\left(h_t^{PS}\right), \sigma_t^{PS,y}\left(h_t^{PS}\right)\right\}$, satisfies the optimality conditions depicted by the NKPC and the DIS (Eqs. (1) and (2)),

$$\sigma_t^{PS,\pi}\left(h_t^{PS}\right) = \kappa \sigma_t^{PS,y}\left(h_t^{PS}\right) + \beta E_t\left[\sigma_{t+1}^{PS,\pi}\left(h_{t+1}^{PS}\right)\right] \tag{40}$$

$$\sigma_t^{PS,y}\left(h_t^{PS}\right) = E_t\left[\sigma_{t+1}^{PS,y}\left(h_{t+1}^{PS}\right)\right] - \left(i_t - E_t\left[\sigma_{t+1}^{PS,\pi}\left(h_{t+1}^{PS}\right)\right] - r_t\right) + \gamma b_t, \tag{41}$$

for all future histories h_{t+1} induced by $\left\{\sigma^{CB}, \sigma^{Tr}\right\}$, and the continuation strategy $\sigma_{t+1}^{PS} \equiv \left\{\sigma_s^{PS,\pi}\left(h_s^{PS}\right), \sigma_s^{PS,y}\left(h_s^{PS}\right)\right\}_{s \geq t+1}$ satisfies

$$\sigma_s^{PS,\pi}\left(h_s^{PS}\right) = \kappa \sigma_s^{PS,y}\left(h_s^{PS}\right) + \beta E_s\left[\sigma_{s+1}^{PS,\pi}\left(h_{t+1}^{PS}\right)\right] \tag{42}$$

$$\sigma_s^{PS,y}\left(h_s^{PS}\right) = E_s\left[\sigma_{s+1}^{PS,y}\left(h_{s+1}^{PS}\right)\right] - \left(\sigma_s^{CB}\left(h_s^{CB}\right) - E_s\left[\sigma_{s+1}^{PS,\pi}\left(h_{s+1}^{PS}\right)\right] - r_s\right) + \gamma \sigma_s^{Tr}\left(h_s^{Tr}\right) \tag{43}$$

for all future histories h_s induced by $\left\{\sigma^{CB}, \sigma^{Tr}\right\}$ with $s \geq t+1$.

For the treasury, standing at any t, the fiscal policy continuation strategy $\sigma_t^{Tr} = \left\{\sigma_s^{Tr}\left(h_s^{Tr}\right)\right\}_{s=t}^{\infty}$ solves the following problem: Given a current history h_t^{Tr} and strategies $\left\{\sigma^{PS}, \sigma^{CB}\right\}$, then the treasury does

$$\max_{\left\{\tilde{\sigma}_k^{Tr,\tau}, \tilde{\sigma}_k^{Tr,b}, \tilde{\sigma}_k^{Tr,\ell}\right\}_{k=t}^{\infty}} -\frac{1}{2} E_t \left\{\sum_{k=t}^{\infty} \beta^{k-t}\left[\alpha_y^{Tr}\left[\sigma_k^{PS,y}\left(h_k, \left(\sigma_k^{CB}\left(h_k^{CB}\right), \tilde{\sigma}_k^{Tr}\left(h_k^{Tr}\right)\right)\right)\right]^2 + \alpha_\tau^{Tr}\left[\tilde{\sigma}_k^{Tr,\tau}\left(h_k^{Tr}\right)\right]^2\right]\right\} \tag{44}$$

subject to

$$\begin{aligned}
\sigma_k^{PS,y}\left(h_k, \left(\sigma_k^{CB}\left(h_k^{CB}\right), \tilde{\sigma}_k^{Tr}\left(h_k^{Tr}\right)\right)\right) \quad &= E_k\left[\sigma_{k+1}^{PS,y}\left(h_{k+1}, \left(\sigma_{k+1}^{CB}\left(h_{k+1}^{CB}\right), \tilde{\sigma}_{k+1}^{Tr}\left(h_{k+1}^{Tr}\right)\right)\right)\right] - \\
&\quad \left(\sigma_k^{CB}\left(h_k^{CB}\right) - E_k\left[\sigma_{k+1}^{PS,\pi}\left(h_{k+1}, \left(\sigma_{k+1}^{CB}\left(h_{k+1}^{CB}\right), \tilde{\sigma}_{k+1}^{Tr}\left(h_{k+1}^{Tr}\right)\right)\right)\right] - \right. \\
&\quad \left. r_k\right) + \gamma \tilde{\sigma}_k^{Tr,b}\left(h_k^{Tr}\right)
\end{aligned}$$

$$\tilde{\sigma}_k^{Tr,b}\left(h_k^{Tr}\right) = \tilde{\sigma}_k^{Tr,\ell}\left(h_k^{Tr}\right) + (1+r)\,\tilde{\sigma}_{k-1}^{Tr,b}\left(h_{k-1}^{Tr}\right) - \tilde{\sigma}_k^{Tr,\tau}\left(h_k^{Tr}\right)$$

$$\begin{array}{lll}
0 < \tilde{\sigma}_k^{Tr,b}\left(h_k^{Tr}\right) \leq \bar{b} < \infty, & \tilde{\sigma}_k^{Tr,\ell}\left(h_k^{Tr}\right) \geq 0, & \tilde{\sigma}_k^{Tr,\tau}\left(h_k^{Tr}\right) \geq 0, \\
\tilde{\sigma}_k^{Tr,\ell}\left(h_k^{Tr}\right) \mathbf{1}_{\{r \neq \underline{r}\}} = 0, & \tilde{\sigma}_k^{Tr,b}\left(h_k^{Tr}\right) \mathbf{1}_{\{r \neq \underline{r}\}} = 0, & \tilde{\sigma}_k^{Tr,\tau}\left(h_k^{Tr}\right) \mathbf{1}_{\{r = \underline{r}\}} = 0,
\end{array}$$

for all future histories h_k induced by $\left\{\sigma^{CB}, \sigma^{Tr}\right\}$ with $k > t$.

Finally for the central bank, standing at any t, the monetary policy continuation strategy $\sigma_t^{CB} = \left\{\sigma_s^{CB}\left(h_s^{CB}\right)\right\}_{s=t}^{\infty}$ solves the following problem: Given a current history h_t^{CB} and strategies $\left\{\sigma^{PS}, \sigma^{Tr}\right\}$, then the CB does

$$\max_{\left\{\tilde{\sigma}_k^{CB,i}\right\}_{k=t}^{\infty}} -\frac{1}{2}E_t\left\{\sum_{k=t}^{\infty}\beta^{k-t}\left[\alpha_y^{CB}\left[\sigma_k^{PS,y}\left(h_k, \tilde{\sigma}_k^{CB}\left(h_s^{CB}\right), \sigma_k^{Tr}\left(h_s^{Tr}\right)\right)\right]^2 + \alpha_\pi^{CB}\left[\sigma_k^{PS,\pi}\left(h_k, \tilde{\sigma}_k^{CB}\left(h_s^{CB}\right), \sigma_k^{Tr}\left(h_s^{Tr}\right)\right)\right]^2\right]\right\} \tag{45}$$

subject to

$$\begin{aligned}
\sigma_k^{PS,\pi}\left(h_k, \left(\tilde{\sigma}_k^{CB}\left(h_k^{CB}\right), \sigma_k^{Tr}\left(h_k^{Tr}\right)\right)\right) &= \kappa\sigma_k^{PS,y}\left(h_k, \left(\tilde{\sigma}_k^{CB}\left(h_k^{CB}\right), \sigma_k^{Tr}\left(h_k^{Tr}\right)\right)\right) + \\
&\quad \beta E_k\left[\sigma_{k+1}^{PS,\pi}\left(h_{k+1}, \left(\tilde{\sigma}_{k+1}^{CB}\left(h_{k+1}^{CB}\right), \sigma_{k+1}^{Tr}\left(h_{k+1}^{Tr}\right)\right)\right)\right] \\
\sigma_k^{PS,y}\left(h_k, \left(\tilde{\sigma}_k^{CB}\left(h_k^{CB}\right), \sigma_k^{Tr}\left(h_k^{Tr}\right)\right)\right) &= E_k\left[\sigma_{k+1}^{PS,y}\left(h_{k+1}, \left(\tilde{\sigma}_{k+1}^{CB}\left(h_{k+1}^{CB}\right), \sigma_{k+1}^{Tr}\left(h_{k+1}^{Tr}\right)\right)\right)\right] - \\
&\quad \left(\tilde{\sigma}_k^{CB}\left(h_k^{CB}\right) - E_k\left[\sigma_{k+1}^{PS,\pi}\left(h_{k+1}, \left(\tilde{\sigma}_{k+1}^{CB}\left(h_{k+1}^{CB}\right), \sigma_{k+1}^{Tr}\left(h_{k+1}^{Tr}\right)\right)\right)\right] - \right. \\
&\quad \left. r_k\right) + \gamma\sigma_k^{Tr,b}\left(h_k^{Tr}\right)
\end{aligned}$$

$$\tilde{\sigma}_k^{CB,i}\left(h_k^{CB}\right) \geq 0.$$

for all future histories h_k induced by $\left\{\sigma^{CB}, \sigma^{Tr}\right\}$ with $k > t$.

It is important to note that the solutions to problems (44) and (45) are sequences of functions (i.e., strategies) for the central bank and the treasury with: (1) Arguments depending on histories, and (2) solutions involving elements dated at future events. As a result, the solutions depend on the history restrictions observed by the optimizing policymakers. For example, we may consider the polar cases of histories based only on current occurrences of the shock (Markov), or histories with some finite memory. Also, the solutions to these problems will vary depending on the policymakers' ability to affect either future expectations for every future period (i.e., for every future element of the sequences forming the policymakers' strategies), or for a limited number of periods.

Having described the strategies for each player and structure of histories, I now define a sustainable equilibrium for the model economy.

Definition 5 *A sustainable equilibrium (SE) of the model (denoting sequences and their corresponding elements with superindex S) is a triple* $\left\{\sigma^{CB,S}, \sigma^{Tr,S}, \sigma^{PS,S}\right\}$ *such that for every history* h_t *the following holds:*

SE1. given $\left\{\sigma^{CB,S}, \sigma^{Tr,S}\right\}$*, the continuation strategy of the private sector,* $\sigma_t^{PS,S}$*, satisfies rational expectations and the NKPC and the DIS (Eqs. (40)-(43)) for every current history* h_t^{PS}*,* *SE2. given* $\left\{\sigma^{PS,S}, \sigma^{Tr,S}\right\}$*, the continuation strategy of the central bank,* $\sigma_t^{CB,S}$*, solves the central bank's problem (45) for every current history* h_t^{CB}*, and*

SE3. given $\left\{\sigma^{PS,S}, \sigma^{CB,S}\right\}$*, the continuation strategy of the treasury,* $\sigma_t^{Tr,S}$*, solves the treasury's problem (44) for every current history* h_t^{Tr}*.*

Intuitively, Definition (5) states that, for any history h_t, then each *continuation strategy* for $k = \{CB, Tr, PS\}$ associated to k's *own strategy* prescribes a best response to the *strategies* from the other actors in the economy. The definition is in the spirit of Chari and Kehoe (1990), and it adapts Kurozumi (2008) to depict sustainable equilibria in the presence of another policymaker – see, for example, Basso (2009). For early applications of sustainable equilibria applied to credible monetary policy, see for example Chari et al. (1998); for more recent examples, see Nakata (2018) and Walsh (2018).

Finally, and before turning to the characterization of sustainable equilibrium from the next section, it will be useful to introduce the following object. Standing at any t, and given a current history h_t and strategies $\left\{\sigma^{PS}, \sigma^{Tr}\right\}$, call

$$\begin{aligned} V_t^{CB,\tilde{\sigma}^{CB}} \equiv & -\frac{1}{2}E_t\left\{\sum_{k=t}^{\infty}\beta^{k-t}\left[\alpha_y^{CB}\left[\sigma_k^{PS,y}\left(h_k,\left(\tilde{\sigma}_k^{CB}\left(h_k\right),\sigma_k^{Tr}\left(h_k\right)\right)\right)\right]^2+\right.\right. \\ & \left.\left.\alpha_\pi^{CB}\left[\sigma_k^{PS,\pi}\left(h_k,\left(\tilde{\sigma}_k^{CB}\left(h_k\right),\sigma_k^{Tr}\left(h_k\right)\right)\right)\right]^2\right]\right\} \end{aligned} \tag{46}$$

where

$$\begin{aligned} \sigma_k^{PS,\pi}\left(h_k,\left(\tilde{\sigma}_k^{CB}\left(h_k^{CB}\right),\sigma_k^{Tr}\left(h_k^{Tr}\right)\right)\right) &= \kappa\sigma_k^{PS,y}\left(h_k,\left(\tilde{\sigma}_k^{CB}\left(h_k^{CB}\right),\sigma_k^{Tr}\left(h_k^{Tr}\right)\right)\right)+ \\ & \beta E_k\left[\sigma_{k+1}^{PS,\pi}\left(h_{k+1},\left(\tilde{\sigma}_{k+1}^{CB}\left(h_{k+1}^{CB}\right),\sigma_{k+1}^{Tr}\left(h_{k+1}^{Tr}\right)\right)\right)\right] \\ \sigma_k^{PS,y}\left(h_k,\left(\tilde{\sigma}_k^{CB}\left(h_k^{CB}\right),\sigma_k^{Tr}\left(h_k^{Tr}\right)\right)\right) &= E_k\left[\sigma_{k+1}^{PS,y}\left(h_{k+1},\left(\tilde{\sigma}_{k+1}^{CB}\left(h_{k+1}^{CB}\right),\sigma_{k+1}^{Tr}\left(h_{k+1}^{Tr}\right)\right)\right)\right]- \\ & \left(\tilde{\sigma}_k^{CB}\left(h_k^{CB}\right)-E_k\left[\sigma_{k+1}^{PS,\pi}\left(h_{k+1},\left(\tilde{\sigma}_{k+1}^{CB}\left(h_{k+1}^{CB}\right),\sigma_{k+1}^{Tr}\left(h_{k+1}^{Tr}\right)\right)\right)\right]-\right. \\ & \left.r_k\right)+\gamma\sigma_k^{Tr,b}\left(h_k^{Tr}\right) \end{aligned}$$

the value of a monetary policy continuation strategy $\tilde{\sigma}_t^{CB} = \left\{\tilde{\sigma}_s^{CB,i}\left(h_s\right)\right\}_{s=t}^{\infty}$ for the central bank, for any future history h_k at periods $k > t$ that is induced by $\{\left\{\sigma_s^{CB}\left(h_s\right)\right\}_{s=t}^{\infty}, \left\{\sigma_s^{Tr}\left(h_s\right)\right\}_{s=t}^{\infty}\}$.

4.2 Sustainable equilibrium characterization

I now proceed to characterize credible FG in the model economy. To that end, I introduce the next Proposition.

Proposition 6 *The No-FG equilibrium is the worst sustainable equilibrium.*

Proof. See proof in Appendix 5. ■

Proposition 6 is part of a three-step approach commonly followed in the literature of sustainable equilibria – see, for instance, Chari and Kehoe (1990), Kurozumi (2008), Basso (2009), and Nakata (2018). Therefore, first, Proposition 6 argues that the discretionary central bank equilibrium from the previous section is also a sustainable equilibrium. Moreover, Proposition 6 argues that it is the worst SE. Next, and as part of the second step, I proceed to define a trigger strategy that uses such discretionary equilibrium as the scenario towards which the economy reverts upon central bank deviation from its promised FG level. In the last step, this strategy will finally be used to characterize credible FG policies.

Credible forward guidance will impact differently on the economy's outcomes, and will be implementable as long as they are credible. To characterize equilibrium outcomes associated to credible forward guidance, I use a trigger strategy equilibrium. Specifically, I describe a trigger strategy first, and then present a proposition that allows me to characterize the equilibrium outcomes associated with this trigger strategy. For simplicity, I use the name *revert-to-discretion* strategy for this trigger strategy, similar to that in Nakata (2018), and denote it with $\left\{\sigma^{CB,fg}, \sigma^{Tr,fg}, \sigma^{PS,fg}\right\}$. Sequence $\left\{\sigma^{CB,fg}, \sigma^{Tr,fg}, \sigma^{PS,fg}\right\}$ is described as follows. The central bank chooses

$$\text{(CB.1)} \quad \sigma_t^{CB,fg}\left(h_0^{CB}\right) = i_0^{fg}\left(h_0^{CB}\right) \text{ for every initial state in } \mathcal{S}$$

$$\text{(CB.2)} \quad \sigma_t^{CB,fg}\left(h_t^{CB}\right) = \begin{cases} i_t^{fg}\left(h_t^{CB}\right), & \text{if } i_k = i_k^{fg}\left(h_k^{CB}\right) \text{ for every } k \leq t-1 \\ i_t^{*}\left(h_t^{CB}\right), & \text{otherwise.} \end{cases}$$

The revert-to-discretion strategy also instructs the treasury to perform the following actions:

$$\text{(Tr.1)} \quad \sigma_t^{Tr,fg}\left(h_t^{Tr}\right) = \begin{cases} \left(b_t^{fg}\left(h_t^{Tr}\right), \tau_t^{fg}\left(h_t^{Tr}\right), \ell_t^{fg}\left(h_t^{Tr}\right)\right), & \text{if } i_k = i_k^{fg}\left(h_k^{Tr}\right) \text{ for every } k \leq t-1 \\ \left(b_t^{*}\left(h_t^{Tr}\right), \tau_t^{*}\left(h_t^{Tr}\right), \ell_t^{*}\left(h_t^{Tr}\right)\right), & \text{otherwise.} \end{cases},$$

Finally, the Private Sector strategy is

$$\text{(PS.1)} \quad \sigma_t^{PS,fg}\left(h_t^{PS}\right) = \begin{cases} \left(y_t^{fg}\left(h_t^{PS}\right), \pi_t^{fg}\left(h_t^{PS}\right)\right), & \text{if } i_k = i_k^{fg}\left(h_k\right) \text{ for every } k \leq t \\ \left(y_t^{*}\left(h_t^{PS}\right), \pi_t^{*}\left(h_t^{PS}\right)\right), & \text{otherwise.} \end{cases}$$

In the discretionary equilibrium of Section 3, the central bank solved its sequential problem in each state: In particular, the central bank chose zero interest rates in the liquidity trap state Z, and it was free to choose nominal interest rates in state R. In that equilibrium, however, the central bank was unable to affect Private Sector expectations. The trigger strategy of this section now instructs the central bank to give up its ability to implement the discretionary solution in state R, while at the same time it gains the ability to make promises about future paths of nominal interest rates that the Private Sector can rationally expect will be followed by the monetary authority. For the Private Sector, its trust on the discretionary CB's promises lasts as long as the CB keeps its reputation of implementing its present and past promises, and it stops trusting the CB otherwise. In particular, when the latter happens, the Private Sector instantaneously expects the discretionary outcome to hold in the future, and this scenario follows in the model economy thereafter.

Equipped with the previous definitions and objects, the next Proposition provides conditions that characterize economy's outcomes associated to the sustainable equilibria that use trigger strategy outlined above.

Proposition 7 *Be an arbitrary set of fiscal and monetary policies, allocations and inflation rates,* $\left(i_t^S, b_t^S, \tau_t^S, \ell_t^S, y_t^S, \pi_t^S\right)$. *This set is the outcome of an SE if and only if for* $t \geq 0$,

1. $\left\{y_t^S, \pi_t^S\right\}$ *is a PSCE,*

2. $\left\{b_t^S, \tau_t^S, \ell_t^S\right\}$ *solves the optimization problem of the treasury, and*

3. $\left\{i_t^S, b_t^S, \tau_t^S, \ell_t^S, y_t^S, \pi_t^S\right\}$ *satisfies the following sustainability constraint (SC)*

$$V_t^{CB,S}\left(i_t^S, b_t^S, \tau_t^S, \ell_t^S, y_t^S, \pi_t^S\right) \geq V_t^{CB,*}\left(i_t^*, b_t^*, \tau_t^*, \ell_t^*, y_t^*, \pi_t^*\right), \tag{47}$$

where $V_t^{CB,S}$ *represents the time-t present discounted value (PDV) of the central bank's losses under the sustainable equilibrium (i.e., under credible FG), and* $V_t^{CB,*}$ *represents the time-t PDV of the central bank losses under the discretionary equilibrium (i.e., in the No-FG equilibrium).*

Proof. See proof in Appendix 5. ■

The first and second requirements call the equilibrium to be a competitive outcome, and to be consistent with the treasury's optimization problem. The last requirement from Proposition 7 prescribes that the central bank will follow an arbitrary path for nominal interest rates as long as that path allows the central bank to attain payoffs that are greater (or at least equal to) the payoffs under discretion. This requirement is particularized by the satisfaction of the Sustainability Constraint (SC for simplicity) given in expression (47). Next, I focus on building the SC constraint.

It is important to note first that the only relevant Sustainability Constraint (Eq. 47) needed to characterize sustainable equilibria in this model is the SC at state R. The reason for this is that, in periods where the state is Z, the central bank has no other choice than to set $i_Z = 0$. Specifically, the shock parameterization will always be large enough to make the nominal interest rate in state Z (absent any ZLB constraint) to become negative. But the CB cannot choose negative nominal interest rates because the ZLB binds. Thus, in period Z, we are assured that the discretionary CB will have no incentives to deviate. Furthermore, if the CB cannot deviate (and absent any deviation in the past), then the private sector can only rationally expect the CB to preserve its promise. Hence, SC only matters for periods where the state is R (and, in that case, the CB can indeed deviate or not).

To build the Sustainability Constraint (Eq. 47), observe first that in the discretionary equilibrium the strategy profile $\left\{\sigma^{CB}, \sigma^{PS}, \sigma^{Tr}\right\}$ is given by $\left\{\sigma^{CB,*}, \sigma^{PS,*}, \sigma^{Tr,*}\right\}$, with actions prescribed by Proposition (4). Now, given these actions, the per-period losses for the central bank under discretion are time invariant in each state of the world. In other words, we can present a recursive version of the discounted value of the losses, $V_t^{CB,*}$, standing at any period t and for each state of the economy. Since we are interested in evaluating the optimal policies that can be sustained during state R (which is the state where the central bank can deviate since the ZLB constraint is no longer binding) we can write the value of the losses for the discretionary central bank at R as follows (the step-by-step procedure is detailed in the Appendix),

$$V_R^{CB,*} \equiv l_R^{CB}\left(y_R^*\left(i_R^*\right), \pi_R^*\left(i_R^*\right)\right) + \beta p \frac{l_Z^{CB}\left(y_Z^*\left(i_R^*\right), \pi_Z^*\left(i_R^*\right)\right) + \beta l_R^{CB}\left(y_R^*\left(i_R^*\right), \pi_R^*\left(i_R^*\right)\right)}{1-\beta^2 p}. \quad (48)$$

Expression (48) particularizes the right-hand-side of the SC. Furthermore, since alternative equilibrium outcomes that we attempt to sustain will depend on forward guidance levels (i.e., on nominal interest rates at state R, i_R^S), then expression (48) has also been written as an explicit function of the discretionary nominal interest rate at R, i_R^*.

Just like we did for the value of the losses for the CB at R under discretion, we build the same value for any arbitrary forward guidance level, i_R^{fg} – and note that I use superscript fg instead of the generic S from the Proposition because we are focusing on our revert-to-discretion strategy. Formally,

$$V_R^{CB,fg} \equiv l_R^{CB}(y_R(i_R^{fg}), \pi_R(i_R^{fg})) + \beta p \frac{l_Z^{CB}(y_Z(i_R^{fg}), \pi_Z(i_R^{fg})) + \beta l_R^{CB}(y_R(i_R^{fg}), \pi_R(i_R^{fg}))}{1-\beta^2 p}. \quad (49)$$

A forward guidance policy the Central bank implements in the Recovery state constitutes a sustainable equilibrium if and only if it satisfies the SC defined as

$$\overbrace{l_R^{CB}\left(y_R(i_R^{fg}),\pi_R(i_R^{fg})\right)+\beta p\frac{l_Z^{CB}\left(y_Z(i_R^{fg}),\pi_Z(i_R^{fg})\right)+\beta l_R^{CB}\left(y_R(i_R^{fg}),\pi_R(i_R^{fg})\right)}{1-\beta^2 p}}^{V_R^{CB,fg}}\geq \\ \underbrace{l_R^{CB}\left(y_R\left(i_R^*\right),\pi_R\left(i_R^*\right)\right)+\beta p\frac{l_Z^{CB}\left(y_Z\left(i_R^*\right),\pi_Z\left(i_R^*\right)\right)+\beta l_R^{CB}\left(y_R\left(i_R^*\right),\pi_R\left(i_R^*\right)\right)}{1-\beta^2 p}}_{V_R^{CB,*}}. \tag{50}$$

The previous result shows that the Central bank can credibly announce (and maintain) some level of nominal interest rate, i_R^{fg}, to hold during the Recovery state as long as the the present value of such policy exceeds the gains from (i) deviating to the nominal interest rate level that solves the sequential problem for the central bank in the Recovery state, i_R^*, and (ii) foregoing the ability to conduct forward guidance in the future. $V^{CB,*}$ represents the payoff the central bank obtains when it forgoes its ability to make promises and it instead optimizes in every period. $V^{CB,fg}$ represents the payoff the central bank obtains when it keeps its ability to control PS expectations and make credible promises (but giving up its discretionary optimal choice).

To gain insight, we can rearrange Eq. (50) to obtain,

$$\beta p\left[l_Z^{CB}\left(y_Z(i_R^{fg}),\pi_Z(i_R^{fg})\right)-l_Z^{CB}\left(y_Z\left(i_R^*\right),\pi_Z\left(i_R^*\right)\right)\right]\geq l_R^{CB}\left(y_R\left(i_R^*\right),\pi_R\left(i_R^*\right)\right)-l_R^{CB}\left(y_R(i_R^{fg}),\pi_R(i_R^{fg})\right). \tag{51}$$

Recall first that the Sustainability Constraint involves the evaluation of gains standing at time R. The right-hand side of Expression (51) then shows the gains the central bank can seize from deviating when the state is R. Specifically, the terms in the RHS indicate the discretionary outcome losses in state R (as captured by $l_R^{CB}\left(y_R\left(i_R^*\right),\pi_R\left(i_R^*\right)\right)$) relative to the forward guidance policy (term $l_R^{CB}\left(y_R(i_R^{fg}),\pi_R(i_R^{fg})\right)$). A positive RHS means that the central bank can seize instantaneous gains from reneging its promise of low nominal interest rates. But, at the same time, the left-hand side shows the (discounted by β) gains the central bank may seize (with probability p) if in state Z the PS believes in the central bank's promises of expansionary monetary policies to be implemented in state R – i.e., FG. Note that these gains that the central bank can attain in Z do not come from the current nominal interest rate (i_t is already at zero), and instead they only arise if the CB can affect PS' expectations. A positive LHS means that, when the Central bank honors its promises in R, then its gains strictly manifest in state Z since the drop in inflation and output gap is mitigated only by the expectation of expansionary monetary policy from the next state, R. In summary, the weak inequality in Expression (51) then highlights that FG is sustainable when the future gains FG yields in state Z is at least as good as the instantaneous gains from deviating from FG.

A final observation that arises from the sustainability constraint is the constraint may hold with strict inequality under some parameterizations. In other words, when the sustainability constraint holds with strict inequality, it characterizes a non-empty range of sustainable forward guidance policies that the central bank can conduct and that are self-rewarding (i.e., deliver outcomes at least as good as the discretionary equilibrium), and enforceable (namely, outcomes that can be sustained under the threat of a perpetual punishment coming from the private sector expectations). This opens the possibility for the existence of multiple equilibria, which together with the relation between credibility of forward guidance and fiscal policy will be the subjects of the next analysis.

4.3 Results

In this section I first ask whether we can characterize credible forward guidance in the presence of fiscal policy. Provided that we can, then we would like to analyze what the impact of the change in fiscal parameters is on the existing levels of forward guidance. Finally, I analyze the welfare impact of forward guidance, and I investigate whether there are relevant policy trade-offs (e.g., complementarity or substitutability between fiscal and monetary policy) that we can identify when a central bank uses reputation-based forward guidance and it faces a fiscal policy that seeks to mitigate a liquidity trap.

4.3.1 Sustainable levels of forward guidance

Parameter γ connects fiscal policy and monetary policy via the private sector's optimality condition represented by the DIS (Eq. 2). As a result, γ captures the impact of fiscal policy in the credibility of forward guidance. Here, I start by showing that we can characterize credible forward guidance policies using Expression (50) for some $\gamma > 0$. Then I discuss how γ affects these policies.

Figure 4 plots the left- and right-hand sides of Eq. (50) for different announcements of nominal interest rates to be implemented during the Recovery state. I use the parameterization from Section 3 which satisfies that equation with strict inequality – for simplicity, subscript R is omitted from these payoffs. The right-hand side of Eq. (50), labeled $V^{CB,*}$, is represented with an horizontal line because in the Recovery state the central bank solves its sequential problem of choosing its best response, which is an interior solution and not a

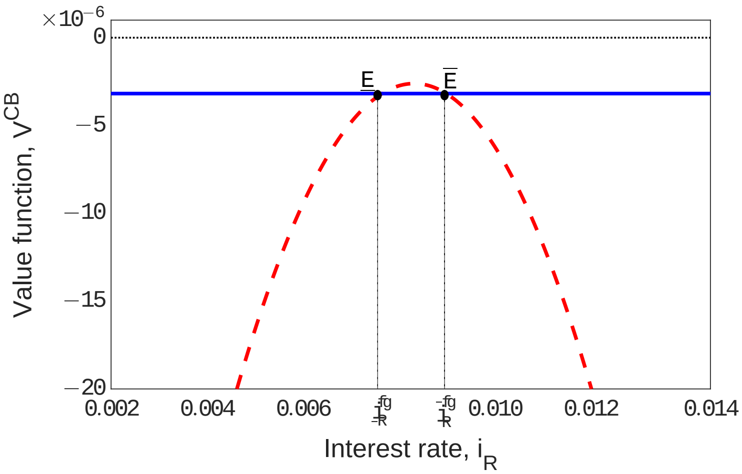


Figure 4: Present discounted value of losses for the central bank. The curves evaluate the central bank's PDV of losses starting at period R and at different nominal interest rates, i_R, when the central bank sustains a policy, $V^{CB,S}$, (dashed red line), and when it implements the discretionary outcome, $V^{CB,*}$, (solid blue line).

function of the announced policies lying on the x-axis. In order to be consistent with the previous section, discretionary results are portrayed with blue solid lines. The left-hand side of the equation, labeled $V^{CB,fg}$, is represented with the red dashed curve, and plots the value of announcing a policy that the private sector believes. Unlike $V^{CB,*}$, value $V^{CB,fg}$ varies with the announcements showing in the x-axis of this graph. This parabola is centered in the range of positive announcements because, in the Recovery state, the natural interest rate is again positive and, hence, the central bank is not constrained to absorb it with its nominal interest rate.

In addition, Figure 4 shows two intersection points, $\underline{E}$ and $\overline{E}$, where the central bank's payoff from sustaining nominal interest rates in state R (i.e., its credible FG) exactly matches the central bank's payoff under discretion. As a result, any forward guidance announcement i_R in the interval $[\underline{i}_R^{fg}, \bar{i}_R^{fg}]$ defines a sustainable equilibrium, each of which is characterized by forward guidance levels chosen from that interval. At $\overline{E}$, the nominal interest rate under discretion, i_R^*, is equal to that under credible forward guidance, $\bar{i}_R^{fg}$, meaning that the central bank is indifferent between either promising a policy that convinces the PS, or choosing a policy that exactly matches its solution to its sequential problem. To the left of point $\overline{E}$, a higher $V^{CB,fg}$ payoff in the open interval $(\underline{i}_R^{fg}, \bar{i}_R^{fg})$ indicates that the central bank can announce lower nominal interest rates in the Recovery state and, thus, exploit credibility gains – instead of implementing the solution to its sequential problem, i_R^*. The intuition for this is the following. Forward-looking expectations of the PS regarding outcomes when the economy is *away from* the liquidity trap can affect outcomes *during* a liquidity trap. But the central bank needs to be credible to have an impact on PS' expectations. If the central bank has credibility, then Figure 4 shows that the central bank may prefer to promise monetary policies more expansionary relative to the discretionary policy (i.e., a nominal interest rate smaller than i_R^*) that mitigates the drop in y_Z and π_Z (at the ZLB) at the cost of distorting y_R and π_R (in the Recovery state).

At point $\underline{E}$ the payoff from implementing credible forward guidance is again equal to the payoff from the policy under no forward guidance (point $\overline{E}$). Here, the credibility gains from forward guidance are exhausted at $i_R = \underline{i}_R^{fg}$ – i.e., below $\underline{i}_R^{fg}$ further distortions in π_R and y_R in the Recovery state cannot compensate the improvements in π_Z and y_Z at the ZLB. To summarize, rates in $[\underline{i}_R^{fg}, \bar{i}_R^{fg}]$ are sustainable, but as we shall see in our welfare analysis in Section 4.3.3, when credibility gains are exhausted at $\underline{i}_R^{fg}$, the present discounted value of welfare will be larger for the central bank and the treasury.

Finally, to the right of $\overline{E}$ or to the left of $\underline{E}$, curve $V^{CB,fg}$ lies below $V^{CB,*}$. To the right of $\overline{E}$, any i_R is more contractionary (i.e., higher) relative to $\bar{i}_R^{fg}$. When nominal rates fall in this range, the central bank is choosing a policy different to its optimal discretionary policy, which can only be reasonable if the central bank is trying to attain some credibility gains. However, with $i_R > \bar{i}_R^{fg}$ the central bank is increasing distortions in Z, which is the opposite that the central bank would optimally seek to do in that state – recall that the central bank is also trying to eliminate these distortions in Z but it cannot only because of the ZLB. Therefore, expanding distortions in Z more by choosing $i_R > \bar{i}_R^{fg}$ goes counter the interest of the central bank, and the private sector rationally anticipates this. In other words, the private sector understands that the central bank does not prefer such contractionary promise, and that the central bank would instead deviate to the discretionary rate – analytically, $V^{CB,fg} < V^{CB,*}$. This means that the central bank cannot guide private sector expectations with rates going in direction $i_R > \bar{i}_R^{fg}$. A simpler logic applies to the left of $\underline{E}$. In this case, it is clear that when $i_R < \bar{i}_R^{fg} (= i_R^*)$ the central bank is attempting to collect gains from credible forward guidance that mitigate the distortions in Z. However, for too expansionary policies, $i_R < \underline{i}_R^{fg}$, the central bank has already exhausted the gains from credible forward guidance because any Z-state improvements cannot offset the large distortions generated in state R. Because of this, such i_R does not survive the rational expectations of the PS: Households anticipate that such a low rate gives the central bank incentives to deviate from i_R to i_R^* and, therefore, the equilibrium with too low rates falls apart.

In summary, if for some parameterization with $\gamma > 0$ the sustainability constraint (Eq. (50)) is satisfied with strict inequality, then the range of nominal interest rates that can be announced, $[\underline{i}_R^{fg}, \bar{i}_R^{fg}]$, is nonempty. Furthermore, this means that we can find an interval $[\underline{i}_R^{fg}, \bar{i}_R^{fg}]$ where the model economy features multiple forward guidance policies that describe different equilibria. Recall first from the No-FG equilibrium that fiscal policy becomes the only game in town during liquidity traps. But if expansionary fiscal policy is costly (because debt has to be repaid in the future), the treasury may not fully offset output drops with too aggressive stabilization policies. After internalizing the treasury's response, the remaining portion of output gap can still give the central bank incentives to mitigate recessions with different levels of reputation-based forward guidance when it has access to it. It is worth pointing out that the previously described mechanisms operating behind credible forward guidance matches expositions from related literature – see, for example, Nakata (2018), Walsh (2018), and Dong and Young (2019).

Before turning to the next section, I shall introduce the following definition that collects all the nominal interest rates that constitute forward guidance announcements that are credible for different parameters $\gamma \geq 0$.

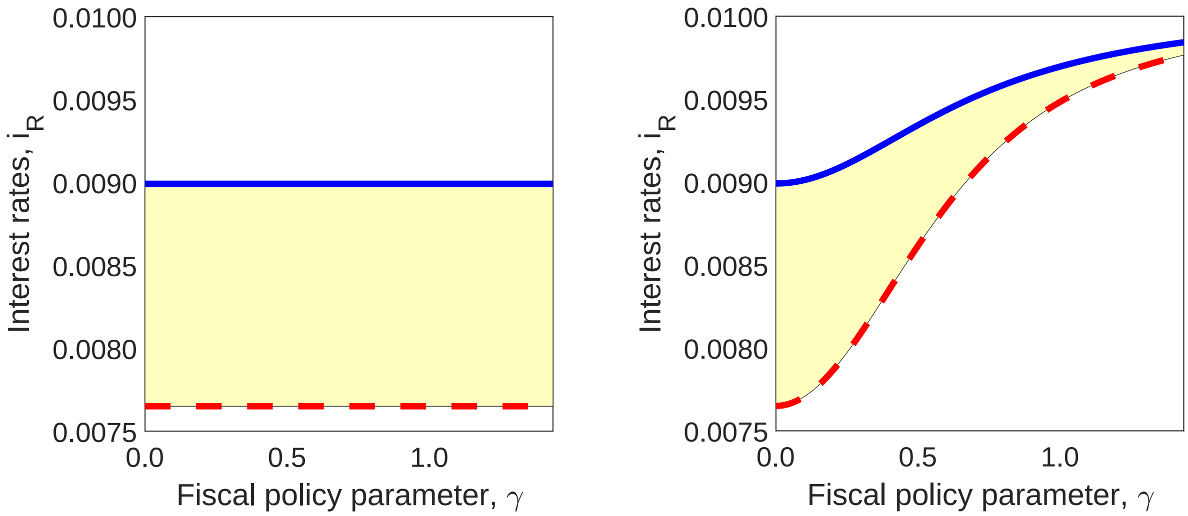
Definition 8 (Credibility Region) *The Credibility Region, $\mathcal{I}$, is the collection of nominal interest rates defined as* $\mathcal{I} \equiv \left\{ i_R^{fg} : i_R^{fg} \text{ satisfies SC (Eq. 51) when } \gamma \geq 0 \right\}$.

This region captures all the nominal interest rates that can be sustained for those γ parameters that satisfy the sustainability constraint at least with equality. Graphically, this region is formed with all the interest rates that lie within intervals $[\underline{i}_R^{fg}, \bar{i}_R^{fg}]$ for some γ (as long as they are non-empty). An example of these intervals is displayed in Figure 4. Having introduced this definition, I now proceed to analyze how region $\mathcal{I}$ responds to changes in fiscal parameters.

4.3.2 Impact of fiscal policy on forward guidance credibility

In this section, I analyze the sensitivity of credible FG to changes in parameter γ. To do so, I will study the effect of γ on region $\mathcal{I}$.

The two panels of Figure 5 show how the presence or absence of fiscal policy impacts on the range of credible forward guidance policies available to the central bank. The impact of fiscal policy in the economy is represented in the horizontal axis, and can be both inactive ($\gamma = 0$) or active ($\gamma > 0$). The forward guidance levels (i.e., the nominal interest rates to be implemented in the Recovery state) appear in the vertical axis. Panel A illustrates how the nominal interest rates i_R^{fg} that lie in the credibility region, $\mathcal{I}$, respond to different fiscal policy parameters (γ) when there is no fiscal policy available in the economy. To remove fiscal policy from the model, observe that setting $\alpha_y^{Tr} = 0$ returns a coefficient $\Phi = 0$ in the treasury's reaction function. Panel B, instead, represents the credibility region where there is fiscal policy available in the economy. The blue solid line plots the highest interest rate that satisfies the sustainability constraint on interest rates (SC) with equality; i.e., $\bar{i}_R^{fg}$. This interest rate corresponds to the case where the central bank announces a forward guidance



(A) FG only.

(B) FG and fiscal policy.

Figure 5: The credibility region. The plot illustrates the region of sustainable forward guidance policies (light-yellow shaded area) for every fiscal policy parameter, γ. The region lies between the highest nominal interest rate that satisfies the sustainability constraint (solid blue line) that coincides with the nominal interest rate at the discretionary equilibrium, and the lowest nominal interest rate that satisfies the SC (dashed red curve).

level, i_R^{fg}, equal to its optimal discretionary policy level, i_R^* – and recall that $\bar{i}_R^{fg} \equiv i_R^*$. The red dashed curve, instead, illustrates the lowest nominal interest rate that satisfies the SC with equality, $\underline{i}_R^{fg}$. The resulting light-yellow shaded area that lies between the blue solid line and the red dashed line shows the region of sustainable (credible) forward guidance policies that the central bank can implement in state R. These are characterized by the vertical intervals of nominal rates between these two contours, at every γ.

Consider first the case of no fiscal policy (Panel A). When $\Phi = 0$, fiscal policy becomes $b_Z = 0$. This indicates that the treasury is not willing to excert fiscal effort in response to monetary policy, output, inflation, or the shock. The left panel of Figure 5 shows that, when debt does not matter, the levels of forward guidance the central bank can promise are unaffected by fiscal policy for every parameter γ. Analytically, the DIS equation shows that changes in the γ parameter are irrelevant to all variables (and, in particular, to monetary policy) when $b_Z = 0$. Then, if $b_Z = 0$, changes in γ should not affect the nominal interest rate that the central bank is willing to sustain in recovery times, and this conjecture is graphically confirmed as the contours of the credibility region $\mathcal{I}$ are invariant to differet values of γ – the yellow region of the left panel of Figure 5 shows that all the forward guidance levels that the central bank can sustain are the same for every γ. This result captures credible policies similar to those characterized in the literature (see, for example, Nakata, 2018).

Panel B shows the case of an economy that has an additional government agency, a treasury, which implements fiscal policy to mitigate a recession. Observe first that, if fiscal policy is inactive ($\gamma = 0$), fiscal policy does not have any impact on inflation rates or output gaps. In particular, when fiscal policy is inactive, the model environment resembles the case where Ricardian equivalence holds (i.e., agents have infinite planning horizons). If the credible forward guidance region is nonempty, we obtain the largest range of sustainable policies – observe the vertical difference between blue and red curves. The intuition is the following. Since $\gamma = 0$ allocates no role for fiscal policy and therefore frees the treasury to act, a portion of the improvement in output gap (that would otherwise occur in each state if $\gamma > 0$) is forgone. As a result of this, the liquidity trap state is at its worst level. But this worsening of the recession actually provides the central bank more leeway to promise a more aggressive expansionary policy (i.e., lower nominal interest rates) and, hence, region $\mathcal{I}$ achieves the widest range.

Panel B also allow us to capture how fiscal policy can erode the credibility of some forward guidance policies. First note that, moving rightward on the horizontal axis, increases in γ indicate that Ricardian equivalence breaks down as active fiscal policy carries a non-negligible effect: When γ increases, the treasury has incentives to implement expansionary fiscal policy (debt-financed lump-sum transfers to the private sector) that helps mitigate recessions. But this happens at the same time forward guidance is also in place. Then, if forward guidance can be made credible without fiscal policy (i.e., if $\mathcal{I} \neq \varnothing$ when $\gamma = 0$), active fiscal policy can restrict the credibility of forward guidance depending on how large the wealth effect of debt-financed fiscal policy is. The implications of fiscal actions on $\mathcal{I}$ can be described by two changes in region $\mathcal{I}$.

The first change brought by fiscal policy is that region $\mathcal{I}$ *shifts upward*. In other words, the shape of $\mathcal{I}$ can be explained via the rising behavior of its upper and lower contours of the region as γ rises. I shall start by explaining the rise in the lower contour as γ increases. Abstracting, for the moment, from forward guidance, we know that a positive γ provides incentives for the treasury to use fiscal policy to mitigate a recession; i.e. to reduce the output gap during a liquidity trap. Therefore, as γ rises, then γb_Z rises, which improves output gap in state Z, and also in the R-state – via the expectations of the Euler Equation. In summary, a rise in γ means that the discretionary scenario (also known as No-FG) improves. Now, since we are explaining what happens under discretion, this description is absent from sustainable forward guidance policies. However, when we consider forward guidance, we know that the central bank uses these discretionary outcomes (that constitute its payoff under discretion) to build reputation and conduct credible forward guidance. Hence, and as a result of the improvement in both Z and R states of the discretionary scenario brought by fiscal policy when γ rises, the rising lower contour of region $\mathcal{I}$ reveals that a higher γ now makes it more difficult for the central bank to sustain too expansionary nominal interest rates promises in

R. The intuition is the following. Assume that $\widehat{i}_R^S$ is the most expansionary forward guidance policy that the central bank chooses when $\gamma = 0$. This policy is sustainable as long as it satisfies the sustainability constraint; i.e., if it is at least weakly preferred to rate i_R^* of the discretionary scenario – scenario the PS uses as a punishment. But with $\gamma > 0$ we just referred above that the discretionary scenario for the central bank improves relative to the $\gamma = 0$ case: If fiscal policy has an impact on the economy, the central bank already seizes gains stemming from active fiscal policy as (i) fiscal policy turns the recessionary Z-state into a milder episode, and (ii) it also improves the R-state scenario via the expectations of such milder Z scenario. This means that the Z-state outcomes (now better because of fiscal policy) can only tolerate milder state-R distortions with forward guidance and,[29] therefore, any forward guidance level that remains credible (call it $\widetilde{i}_R^S$) must be more contractionary (i.e., $\widetilde{i}_R^S \in (\widehat{i}_R^S, i_R^*)$).[30] In search of that $\widetilde{i}_R^S$, the sustainability constraint rules out rates of the type of $\widehat{i}_R^S$ by moving upward in the y-axis and until the first one that gives no incentives for the central bank to deviate is reached. This eventually shifts up the lower contour of region $\mathcal{I}$.

The upper contour of $\mathcal{I}$ also shifts up. Recall first that the nominal interest rate in the upper contour of the region behaves in the exact same form as the nominal interest rate under discretion – formally, $\bar{i}_R^{fg}$ and i_R^* match. As γ rises, we argued before that the discretionary scenario sees an improvement in both outcomes from states Z and R. And this improvement becomes particularly relevant in state R, as it allows the central bank to change its optimal response. Recall first that fiscal policy mitigates the negative impact of the shock in every state (again, the shock affects the Z state directly, and state R indirectly via expectations). And as these better outcomes in Z and R are seized by the central bank, then the monetary authority can adjust its optimal choice in R and absorb an even larger portion of the natural interest rise with the nominal interest rate; i.e., rise i_R^*. Analytically, it is straightforward to show that the response of the discretionary nominal interest rate in state R is monotone

[29] In other words, any improvement of the discretionary scenario (and, specifically, of its ZLB outcomes when fiscal policy has a positive impact) limits the credibility gains that a central bank using reputation-based forward guidance can exploit in R.

[30] Analytically, these gains can be seen by a rise in $l_Z^*(\cdot)$ and $l_R^*(\cdot)$ in Expression (51) since these losses become now closer to 0 for $\gamma > 0$. Specifically, we know that as γ rises, the satisfaction of Expression (51) is more difficult because

$$\beta p[l_Z^{fg} - \underset{\uparrow}{l_Z^*}] \geq \underset{\uparrow}{l_R^*} - l_R^{fg}.$$

Recall the numerical result in Figure 3 that shows the improvement of discretionary outcomes as γ increases. If for such $\gamma > 0$ the interest rate $\widehat{i}_R^S$ (the rate the central bank sustained when $\gamma = 0$) turns out to be now too expansionary relative to those higher $l_Z^*(\cdot)$ and $l_R^*(\cdot)$ losses attained at some $\gamma > 0$ in the discretionary setup (i.e., where fiscal policy is already mitigating the losses in both states), then it means that $\widehat{i}_R^S$ is no longer sustainable – Expression (51) would exhibit the opposite (and strict) inequality evaluated at $\widehat{i}_R^S$ when $\gamma > 0$.

increasing to γ, namely,

$$\frac{di_R^*}{d\gamma} = \left\{ -\left[\alpha_y^{CB} \left(\left(1 - \left(\beta^{PS} \right)^2 p \right) + \kappa \right) + \alpha_\pi^{CB} \kappa^2 \left(1 + \kappa + \beta^{PS} \right) \right] p \alpha_\tau^{Tr} \beta^{Tr} \left(1 + r_R \right)^2 \frac{r_Z}{\phi^2} \right\} \frac{d\phi}{d\gamma} > 0, \tag{52}$$

where the term in braces is positive, and also $d\phi/d\gamma > 0$ – the positive sign of the derivative follows from the negative term and a negative r_Z. Derivative (52) shows that when fiscal policy becomes more active (γ rises), the nominal interest rate under discretion, i_R^*, is further away from the zero-lower bound in state R. Intuitively, a larger γ releases part of the pressure that the monetary authority faces in the Recovery times since state-Z deflation affects state-R variables via expectations.

Additionally, a larger impact of fiscal policy also contributes to a *reduction* of the region $\mathcal{I}$, which is reflected in Figure 5 via the shrinking in the credibility region $\mathcal{I}$ for each $\gamma > 0$. This behavior is explained by the upward shift in the contours that we described above, in combination with the sign of the derivative $di_R^*/d\gamma$ and an upper bound that the upper contour of $\mathcal{I}$ has. Specifically, as fiscal policy mitigates recession and deflation in state Z, it liberates the pressure on the central bank's nominal interest rate i_R^* to absorb more of rate r_t in state R. But, on the other hand, this rise in i_R^* is actually limited by $\bar{r}$.[31] Therefore, the combined effect of FG announcements dropping out of region $\mathcal{I}$ (which rise the lower contour of $\mathcal{I}$), and the rise of i_R^* (which rises the upper contour of $\mathcal{I}$ and has the distinctive feature of being limited by $\bar{r}$), explains that region $\mathcal{I}$ shrinks as γ rises.

A key finding that can be drawn from this experiment is that these two effects of fiscal policy on monetary policy (namely, the upward shift and the shrinking of the range of state-R nominal interest rates) together imply that active fiscal policy can potentially crowd out some monetary policy promises, or in other words, erode the credibility of reputation-based forward guidance.

Some comparative statics. Figure 6 plots the response of the nominal interest rate in state R to changes in the treasury's weight on tax distortions. The figure shows the lowest (i.e., more expansionary) and the highest (or discretionary) FG levels that satisfy the SC with equality using a red-dashed and a blue-solid line, respectively. The pink shaded region denotes the FG levels where the SC is satisfied with strict inequality. As can be seen in Figure 6, for large concerns over tax distortions (large α_τ^{Tr}), the vertical distance between

[31] From derivative (52) we note that i_R^* increases as γ rises. Furthermore, it is easy to see from the expression of i_R^* in Proposition 4 that i_R^* asymptotically approaches r_R as γ rises (see Appendix). However, note that taking this limit can make $i_Z^* = 0$ become no longer a corner solution but an optimal solution for some positive γ (where the multiplier of the ZLB becomes 0, and the liquidity trap is no longer binding). The ZLB multiplier in state Z in our analysis, however, is positive for the γ values considered.

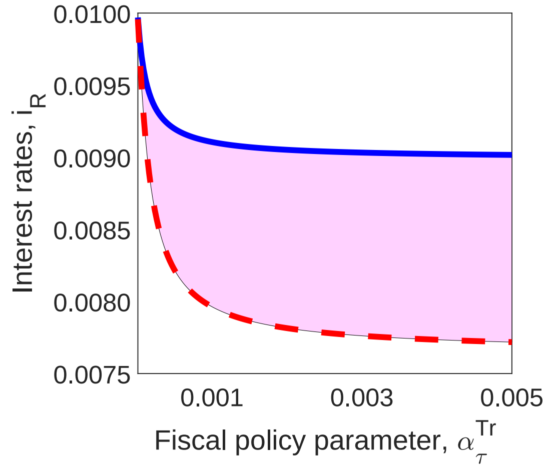


Figure 6: The credibility region. The plot illustrates the region of sustainable forward guidance policies (pink shaded area) for every fiscal policy parameter, α_τ^{Tr}. The region lies between the highest nominal interest rate that satisfies the sustainability constraint (solid blue line) that coincides with the nominal interest rate at the discretionary equilibrium, and the lowest nominal interest rate that satisfies the SC (dashed red curve).

the lower-contour and upper-contour of the region characterizes an interval of nominal interest rates that constitute credible FG policies for the central bank. Therefore, when the treasury's concerns over tax distortions decrease (i.e., moving leftward on the x-axis), there is a reduction in the range of credible FG – which is captured by the shorter height of the pink region for each α_τ^{Tr}.

A similar reasoning to the one in Figure 5 can be used to interpret the response of FG observed here. From the treasury's F.O.C. we know that, given some monetary policy and private sector expectations, the best response to a shock, b_Z, is inversely related to concerns over tax distortions – namely, a decrease α_τ^{Tr}. The logic behind the rise in the lower contour of the region is straightforward. First, a decrease in α_τ^{Tr} making policy b_Z more expansionary means that the treasury is more actively implementing fiscal stabilization policies at the ZLB. But, then, this makes some nominal interest rates not sustainable, with a same argument as before: A smaller α_τ^{Tr} improves Z and R outcomes, and at the current FG level the PS rationally expects a deviation from the monetary authority. Therefore, the SC is negative, unless the central bank announces less expansionary policies. This rises the lower contour of the region. When α_τ^{Tr} decreases, also the upper contour also shifts up. As we argued above, this is because the solution under discretion faces less pressure in state R when fiscal policy improves the Z state (here due to the decrease in α_τ^{Tr}). Finally, the upward shift of the contours, together with the existence of r as an upper bound to how far the state-R discretionary nominal interest rate can rise explains the shrinking of the region in Figure 5.

4.3.3 Welfare analysis and forward guidance-fiscal policy interaction

Thus far, we have assessed the sustainability of forward guidance while it interacts with fiscal policy. Also, we analyzed the effects of fiscal policy on forward guidance for different parameterizations of γ. It would be reasonable to ask, then, how these policy responses affect welfare and equilibrium allocations. In this section, I analyze how welfare differs between the two different monetary policy regimes (one with no forward guidance, and one with forward guidance), in combination with fiscal policy.

We first need to define a measure that captures the welfare impact of the equilibrium allocations. Note, however, that a direct measure of social welfare cannot be easily derived as we have two policymakers with differing objectives. To overcome this issue, I shall consider a convex combination of the discounted losses standing at Z of both fiscal and monetary authorities and call it the social (i.e., private sector) losses. To that end, I will first call the present discounted value (PDV) of losses for authority $k = \{CB, Tr\}$ at time Z under regime $j = \{*, fg\}$ as

$$V_Z^{k,j} \equiv \frac{l_Z^{k,j} - \beta l_R^{k,j}}{1 - \beta^2 p}.$$

It is straightforward to see that the gains from forward guidance are relatively larger than those under discretion for authority $k = \{CB, Tr\}$ if $V_Z^{k,fg} \geq V_Z^{k,*}$. Having defined these PDV of losses standing at Z for the treasury and the central bank, I now define the social (or private sector) losses under regime $j = \{*, fg\}$ as

$$V_Z^{PS,j} = \theta V_Z^{CB,j} + (1 - \theta) V_Z^{Tr,j}$$

where $\theta \in [0, 1]$ captures the relevance of central bank's policy objectives relative to those of the treasury. When $\theta = 1$, private sector losses exactly match those of the central bank. In this case, the private sector allocates no weight to tax distortions, and it instead values output gap and price stability according to the central bank's mandates over these variables. On the contrary, the polar opposite case, $\theta = 0$, means that the private sector has no concerns over price stability, and it only cares about close-to-zero output gaps and low taxation costs. Finally, for any $\theta \in (0, 1)$, social preferences exhibit some fraction of dislike for output gap distortions, inflation, and taxation costs. Figure 9 in the Appendix shows the hump-shaped plot of both central bank and treasury value functions, which is a result of the quadratic structure of the losses in their response to forward guidance announcements from the central bank. Similarly to the description in Walsh (2018), these curves highlight how credible forward guidance operates by affecting private sector expectations. In a nutshell, the decreasing side of the curves shows that lowering nominal interest rates can enhance welfare via state-Z outcomes that are improved at the expense of distortions in R – the opposite logic applies if nominal interest rates are in the increasing side of the curves.

For our current parameterization, the range of sustainable policies for every $\gamma \geq 0$ is in the decreasing side of the curves. This means that the central bank can be better-off by promising lower nominal interest rates. The implications of this for the treasury and for different γ parameters will be further discussed in the table below.

Fiscal parameter: $\gamma = 0.25$					
			Welfare†		
	i_R	b_Z	Central bank	Treasury	Social‡
No-FG	0.0091	0.0022	−1.27	−1.42	−1.35
FG	0.0079	0.0015	−0.75	−0.83	−0.79
Fiscal parameter: $\gamma = 0.50$					
			Welfare†		
	i_R	b_Z	Central bank	Treasury	Social‡
No-FG	0.0093	0.0034	−0.70	−1.05	−0.87
FG	0.0086	0.0028	−0.52	−0.76	−0.64

Table 2: Welfare. Nominal interest rate set at FG lowest level. †: Welfare units multiplied by e^5 to simplify comparison. ‡: Parameter θ set at 0.5 (balanced losses).

Table 2 presents central bank, treasury, and social welfare for two different parameterizations of the fiscal parameter, γ, and under different specifications of monetary policy. Specifically, for each parameterization γ, the table evaluates welfare when the central bank does not have access to forward guidance (row 'No-FG'), and when it can access forward guidance (row 'FG'). When the central bank can implement forward guidance, I set i_R at it most expansionary (i.e., lowest) level, $\underline{i}_R^{fg}$, which is the optimal sustainable forward guidance policy – as measured by the low welfare that it yields the central bank. The first two columns of table 2 display the monetary and fiscal policies, with and without central bank access to forward guidance, and each evaluated at the corresponding fiscal parameter. The last three columns report the welfare corresponding to the central bank, the treasury, and the private sector (labelled 'Social' for brevity). For simplicity, parameter θ has been set at 0.5.

I shall start comparing the impact of γ in the No-FG scenarios beginning with the treasury – see column 4 in the No-FG row for both the top and the lower panels. When fiscal policy is more effective to mitigate a recession during liquidity traps (i.e., fiscal policy has a higher wealth effect due to a larger γ), then the treasury is willing to improve Z outcomes. This can be observed by the increase in b_Z between No-FG levels when γ rises: As lump-sum transfers increase, the amount of debt issued to finance them rises – which rises output in the Euler Equation. Also, note that a higher impact of fiscal policy in state Z affects R-outcomes. Specifically, when fiscal policy has more power to prevent a recession, the improved variables at Z spill over R outcomes due to forward-looking inflation and output gap. As a result, R-outcomes improve too when γ rises, and all these effects combined rise treasury's welfare.

For the central bank, recall first that it always observes better outcomes in Z when fiscal policy is active, and these outcomes can be even better as fiscal policy has more effect; namely, γ rises. Likewise, even when the central bank has no access to forward guidance, it will still seize better outcomes in R because of the impact of state-Z fiscal policy on state-R variables via forward-looking variables. This result is magnified as γ rises, which shows in the larger central bank welfare under No-FG as we switch from the top panel to the bottom panel. Moreover, the improvement in R outcomes explains the rise in the nominal interest rate (column 1) under discretion (No-FG). Specifically, better R-state outcomes allow the central bank to use its nominal interest rate to absorb a higher portion of the natural interest rate, r_t. This explains the rise in i_R when we transition between No-FG when γ rises. In summary, the general effect that a higher impact of fiscal policy has in the economy is that welfare under No-FG for both the Treasury and the Central Bank (and, therefore, social welfare) can improve with credible forward guidance (columns 3 to 5).

I now turn to analyze what happens when the central bank uses credible forward guidance, starting with the case of a fixed γ. When the monetary authority has access to forward guidance, it can lower the nominal interest rate to more expansionary levels when the economy is away from a liquidity trap. For example, when $\gamma = 0.25$ (top panel), i_R drops from its discretionary level, 0.0091, to a FG level of 0.0079 – a 13 percent decline. Now, let us focus on central bank's welfare. On the one hand, recall that the No-FG scenario represents the highest welfare the central bank achieves when it lacks access to reputation. But, on the other hand, this No-FG scenario also constitutes the punishment whenever the CB deviates from an announced forward guidance policy. For this latter case, when the central bank builds reputation to lower i_R from the discretionary level, it tolerates some distortions in y and π in state R as long as these reduce the dispersion in output gap and inflation in Z. Eventually, these R-distortions are at their maximum level when $i_R = \underline{i}_R^{fg}$, and that is where the central bank achieves lower losses and larger welfare. Finally, as γ rises, forward guidance *can* improve welfare further: While No-FG welfare rises as γ rises (i.e., the punishment scenario), then if reputation-based forward guidance is still sustainable for some higher γ, it must be that the gains associated to such forward guidance level yields at least a welfare higher to the (now improved) punishment No-FG scenario.[32]

[32] However, as γ rises, it is more difficult to sustain forward guidance: Numerically, when $\gamma = 0.5$, the range

I now consider fiscal tools. As we discussed above, a positive fiscal parameter γ makes the treasury willing to increase fiscal effort to mitigate recessions. Now, for any given γ, the change in regime from No-FG to FG generates less fiscal effort (as observed in column 2). Recall that, regardless of the access to forward guidance, good or bad prospects about future y and π affect the response of b_Z via private sector's expectations, y_Z^e and π_Z^e. But when the central bank has access to forward guidance, the monetary authority can affect these expectations. Specifically, if the central bank can convince the private sector about implementing a policy rate different than the discretionary rate, then the Treasury issues lower lump sum transfers during Z – which have a lower impact in the economy as measured by γb_t in the DIS. Hence, forward guidance makes the central bank to be in charge of a higher share of the fight against a recession in Z, with lower fiscal action from the treasury's side. With respect to welfare, the treasury is better-off when the central bank uses forward guidance, and the intuition for this result is straightforward. With forward guidance in place, monetary policy affects PS's expectations, and these impact on the reaction function of the treasury. The treasury is better-off because the central bank is now fighting the recession in Z: As forward guidance makes the treasury retire some of its fiscal assistance, the treasury suffers less distortions in R (distortions that arise for the treasury as it needs to collect less taxes to pay its issued debt). To summarize, when the central bank can conduct forward guidance, not only the central bank does better (as compared between the FG rows of the central bank's welfare), but also the treasury can do even better (comparing FG rows of treasury's welfare) because forward guidance crowds out a portion of fiscal effort, thus lowering the amount of tax distortions that it will generate in R for the treasury – and which are costly to the treasury. Finally, all these effects on treasury's welfare are magnified when γ rises (and if some forward guidance levels are still sustainable).

To better explore how fiscal and monetary policy interact when the central bank seeks to implement credible forward guidance while the treasury conducts fiscal policy, I compare equilibrium outcomes when the economy transitions between different levels of forward guidance.

Figure 7 depicts the equilibrium dynamics when the central bank cannot use forward guidance (blue solid lines) versus the equilibrium outcome where the central bank implements the lowest forward guidance policy (red dashed lines), which satisfies the SC and gives the highest welfare. For reference, the curves under no forward guidance are the same as those from Figure 3. The figure helps us to compare the different outcomes that can be attained when the central bank exhausts its credibility power to set the most expansionary level of FG. First of all, note that there are some similarities between the two curves. Again, the negative

of policies is $|0.0086 - 0.0093| = 0.0007$, while for $\gamma = 0.25$, the range of policies is $|0.0079 - 0.0091| = 0.0012$.

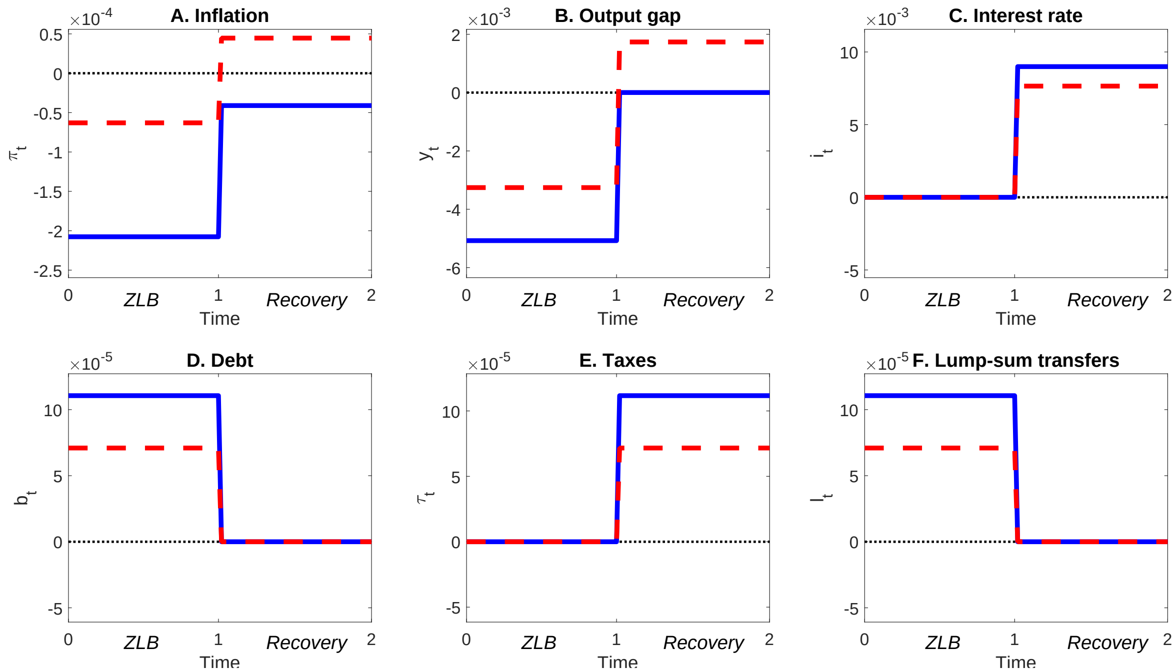


Figure 7: Simulated time series with forward guidance. The lines represent the responses of equilibrium outcomes to a one-period shock to the natural interest rate, for the case with forward guidance (dashed red lines) and without forward guidance (solid blue lines) – in both cases fiscal policy is active, $\gamma > 0$. Variables in log deviations (except i_t).

demand shock triggers an immediate response from fiscal and monetary authorities. Panel C shows that the nominal interest rate curves overlap in both equilibria at the time of the shock (state Z). This is because the nominal interest rate strategy the central bank follows instructs the monetary authority to choose the same rate that the central bank would choose under discretion. Also, in state Z there is a drop in the output gap and a large deflation, which are mitigated with expansionary fiscal policy financed with debt.

The dynamics of the model economy in Figure 7 also evidences the mechanism that the central bank uses to make FG credible (described in the previous section). Specifically, forward guidance offsets costs and gains between states Z and R, and this is channeled by the expectations of the private sector, which make the impact of credible monetary expansions in R states to cascade back into Z states (and viceversa). Credible FG therefore results in milder recessions (or even positive output gaps), and mitigated deflation in both states. Panels A and B show that credible future promises of loose monetary policy during economic recoveries (i.e., lower interest rates in state R) can affect the economy's outcomes in state Z. Overall, then, forward guidance alleviates deflation and recession during liquidity traps: The private sector anticipates that it will have less purchasing power in state R (due to a looser monetary policy when the liquidity trap ends), and it therefore prefers to consume more today – rising y_Z and π_Z, which lowers central bank losses. Therefore, when the central bank can use its reputation, it exploits credibility gains and it can impact the economy using FG via (i) a direct effect in state-R outcomes where π_R and y_R increase, and (ii) an indirect effect in state-Z outcomes, as the PS rationally expects that state-R monetary policy will be delivered. To summarize, the first conclusion from Panels A and B in Figure 7 is that credible FG can mitigate deflation (at Z and R) and recessions (at Z).

I now turn to the analysis of fiscal policy under the presence of forward guidance. Under both discretionary monetary policy and forward guidance, Panel D shows that the treasury implements an expansionary fiscal policy to counter for part of the drop in output gap. Note, however, that Figure 7 shows a change in the direction of the fiscal response compared to the scenario without forward guidance. To see analytically why this is the case, I revisit the treasury's best response from Section 3,

$$b_Z = \frac{\alpha_y^{Tr}\gamma i_Z}{\alpha_y^{Tr}\gamma^2 + \beta^{Tr}\alpha_\tau^{Tr}\left(1+r_R\right)^2} - \frac{\alpha_y^{Tr}\gamma\left(y_Z^e + \pi_Z^e\right)}{\alpha_y^{Tr}\gamma^2 + \beta^{Tr}\alpha_\tau^{Tr}\left(1+r_R\right)^2} - \frac{\alpha_y^{Tr}\gamma r_Z}{\alpha_y^{Tr}\gamma^2 + \beta^{Tr}\alpha_\tau^{Tr}\left(1+r_R\right)^2}.$$

Recall that the treasury's response takes as given the actions from the central bank and PS expectations. Abstracting momentarily from the role of expectations, observe first that the treasury exerts the same positive level of effort upon a negative shock, $-\alpha_y^{Tr}\gamma r_Z/(\alpha_y^{Tr}\gamma^2 + \beta^{Tr}\alpha_\tau^{Tr}(1+r_R)^2)$. But, in fact, with credible FG, the central bank can now affect future y_R and π_R outcomes, which are variables considered by the treasury via the $(y_Z^e + \pi_Z^e)$ term. Therefore, since the treasury best responds not only to the central bank's nominal interest rate, but also to the expectation of future inflation and output gap (π_Z^e and y_Z^e), then the PS' expectation of an improvement in future outcomes coming from credible forward guidance eventually releases the pressure of the treasury to respond with the (large) expansionary fiscal policy observed in Section 3. This implies that, during a recession, any sustainable forward guidance level i_R^{fg} such that $i_R^{fg} < i_R^*$ can make fiscal responses to go in the opposite direction to monetary expansions – i.e., the treasury can now exert a more contractionary fiscal policy when the central bank uses credible FG. This finding provides a new policy lesson, and it reveals that reputation based forward guidance implemented during liquidity traps can yield a 'loose monetary-tight fiscal' policy mix.

To further expand on why there can be responses of fiscal and monetary policy in opposite directions during a recession, I turn to a graphical analysis that plots how the variables of the economy respond to credible forward guidance.

Figure 8 illustrates the impact of different levels of credible forward guidance (nominal interest rate i_R^{fg} in the x-axis) on equilibrium outcomes (y-axis). In particular, Plots E and B show that during Recovery times (resp. ZLB times), more aggressive forward guidance can increase the output gap (resp. mitigate a recession). In particular, note that (i) output in

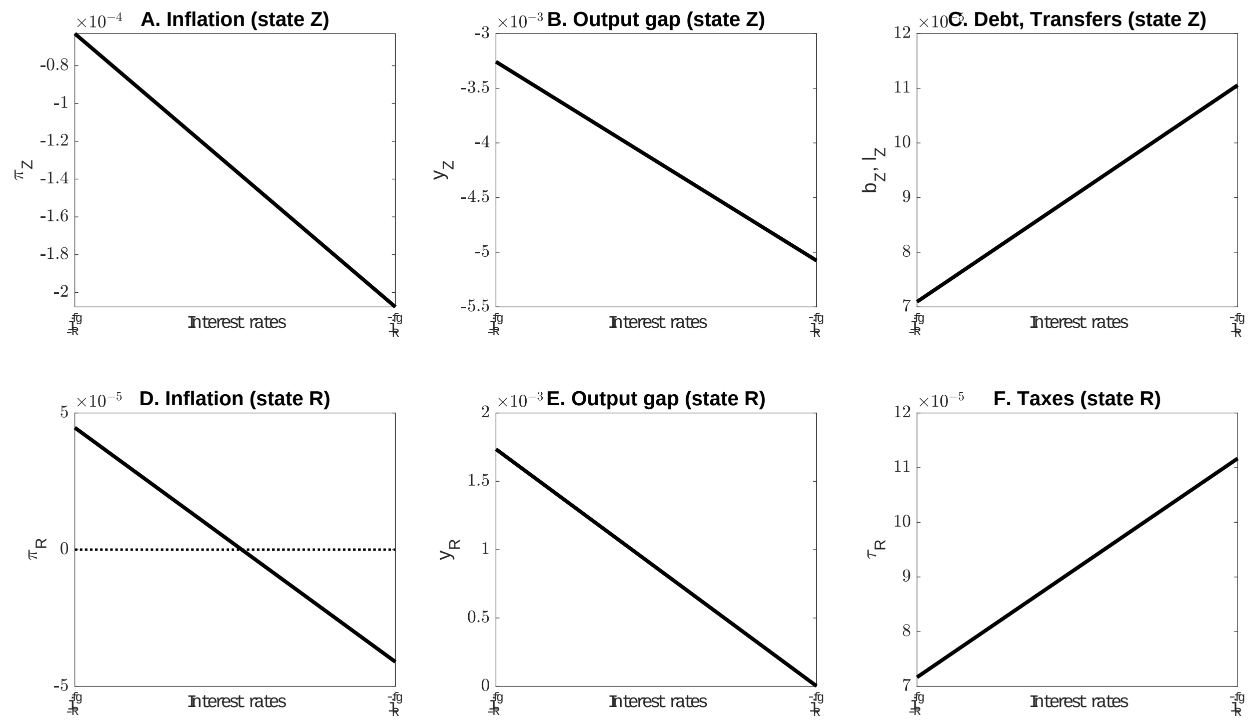


Figure 8: Equilibrium outcomes and FG. The lines show the response of equilibrium outcomes to a one-period shock to the natural interest rate for different levels of sustainable forward guidance (solid lines). Variables in log deviations (except i_t).

the R state collects both the direct effect (coming from the contemporaneous rise in π_R and y_R), and indirect effects (as the PS anticipates an improvement of state-Z outcomes), and (ii) output in the liquidity trap state is benefited by the indirect effect of PS expectations alone (the PS anticipates credible FG in state R). Also, inflation in states Z and R (resp. Plots A and D) show that a more expansionary forward guidance (i.e., as i_t moves leftward in the x-axis) can mitigate deflation in both states. Furthermore, since during liquidity traps the nominal interest rate cannot go below zero, the observed improvement in deflation in state Z stems purely from the monetary policy's effect on the expectations of the private sector. This result coincides with what the literature calls the *deflation bias in expectations* – see Nakov (2008). During the Recovery state, the possibility of future deflation from state Z affects π_R. Thus, inflation can be negative. However, Panel D in Figure 8 shows that forward guidance can counter deflation in R times via both its direct effect (the decrease in i_t), and its indirect effect (less deflation in state Z). In fact, in the R state levels that are closer to the lowest forward guidance equilibrium, $\underline{i}_R^{fg}$, can actually revert what Eggertsson (2006) calls the *deflation bias of discretionary policy.*

So far I have focused on all outcomes except from fiscal policy. In particular, Plot C shows the response of debt and lump sum transfers during a liquidity trap. At the ZLB, a key feature in Plot C is that a more aggressive forward guidance can result in smaller lump-sum transfers. This smaller lump-sum transfers translate into small debt issuance, which carry a smaller impact in the Dynamic IS curve. In other words, forward guidance may trigger a more austere fiscal policy as it crowds out a portion of fiscal effort. To understand this, it is useful to recall that the central bank leads the treasury. Thus, after the monetary authority has set a policy, the treasury observes it and acts accordingly. The key intuition behind it is, again, the way the treasury best responds to forward guidance. In Section 3 we saw that the treasury issues debt in state Z based on private sector's expected inflation and output gap, as well as current monetary policy (set at 0 during the liquidity trap) and the magnitude of the shock. But now the central bank has access to forward guidance that affects PS' expectations. This makes the treasury follow a more conservative fiscal expansion as forward guidance is now improving one of the variables the fiscal authority observes (the output gap). As a result of this, the treasury has now more incentives to implement less expansionary fiscal policies while the central bank attempts to implement FG.

4.3.4 Discussion

In search of linkages between fiscal policy parameters and the credibility of reputation-based forward guidance, this last section additionally showed how the quantitative experiment performed in this paper can give rise to a scenario where both central bank and treasury's policy choices can generate a 'loose monetary-tight fiscal' mix in equilibrium during a recession. In other words, this last result implies that expansionary fiscal policy and reputation-based forward guidance can exhibit a substitutability component while a central bank and a treasury try to fight a crisis. Given that this paper features a stylized model economy that analyzes existing links between one type of unconventional monetary policy (forward guidance) and one type of impact of fiscal policy (wealth effects of debt from fiscal expansions), this last conclusion regarding the substitutability of fiscal and monetary policy during recessions should be treated with caution. However, this finding is relevant since it accounts for the possibility of a lack of coordination between fiscal and monetary policies. In recent years, this has been empirically analyzed by Greenwood et al. (2014), who documented fiscal and monetary policies going in opposite directions with regards to debt management in the US during (and after) the financial crisis of 2008. Furthermore, such disconnect between fiscal and monetary policy can also be traced back to some narrative evidence (as the one below from Mario Draghi when he was president of the European central bank; see Draghi (2019)) which has raised questions about whether monetary and fiscal policy respond cooperatively to crises:

> "[F]iscal policy should play its role. Over the last 10 years, the burden of macroeconomic adjustment has fallen disproportionately on monetary policy. We have even seen instances where fiscal policy has been pro-cyclical and countered the monetary stimulus."

However, and with regards to the analysis of reputation-based forward guidance, this paper constitutes a first attempt that captures policy implications that conflict with the notion of coordinated monetary and fiscal responses during liquidity trap episodes.

In summary, it is worth emphasizing that this result opposes to the conventional wisdom that suggests that fiscal and monetary responses be expansionary when fighting a crisis. Instead, in the light of the numerical exercise conducted in this paper, policies that may seem "misaligned" can instead be interpreted as optimal responses arising from a central bank and treasury interacting to prevent recessions and deflation. This is in line with Eggertsson (2006) who introduced the notion that bad outcomes during liquidity traps can be a consequence of 'policy constraints and inability to commit'. The last finding in this paper, namely the substitutability between fiscal and monetary policy, therefore expands Eggertsson's latter idea (i.e., the role of policy constraints and lack of commitment) for the case of two separate government agencies.

5 Conclusion

This paper addresses the following two questions: (1) How does the credibility of forward guidance respond to active fiscal policy? (2) What are key features arising from the interaction between credible forward guidance and active fiscal policy? To answer to these questions, I use a standard new Keynesian environment adjusted to capture only wealth effects from debt-financed fiscal policy. I outline two separate government agencies, a treasury and a central bank, that use fiscal and monetary policy respectively to mitigate recessions and deflation during repeated liquidity trap states triggered by negative shocks to the natural interest rate.

When the central bank's nominal interest rate hits the zero-lower-bound and this is the central bank's only instrument, I show that the treasury uses expansionary fiscal policy to mitigate a portion of the deflation and recession that the economy features during a liquidity trap. If, in turn, we allow a central bank to use the repeated structure of liquidity traps to implement FG, I find that multiple FG promises can be made credible. When credible, FG can make private sector's expectations of states with improved outcomes to affect bad outcomes of the liquidity trap period. While this result is in line with the literature, a new key finding is that fiscal policy parameters can restrict the credibility of forward guidance by reducing the range of announcements that can be sustained. This is a novel result that adds to the literature of credible forward guidance and highlights how this monetary tool might be restricted with active fiscal policy. Finally, the simple modelling approach taken in this paper documents that fiscal and monetary policy can depict substitutability even during a crisis: When the economy transitions from an equilibrium without forward guidance to one with forward guidance, the use of this unconventional monetary policy tool can crowd out fiscal effort. In this context, this simple model proves well-suited to explain seemingly misaligned fiscal-monetary policies that emerge in equilibrium.

The majority of the findings presented in this work correspond to a quantitative analysis performed on a simple model that is rich enough to represent credible forward guidance, and to illustrate how fiscal policy may undermine the use of that monetary policy instrument by eroding its credibility. However, some candidate extensions emerge as natural questions: For instance, the role of seigniorage revenues, the impact of the stochastic duration of ZLB episodes, to name a few. These are examples of questions not answered in this paper, but which I plan to explore in future research.

References

Adam, K. and R. M. Billi (2006). Optimal monetary policy under commitment with a zero bound on nominal interest rates. *Journal of Money, credit and Banking*, 1877–1905.

Adam, K. and R. M. Billi (2008). Monetary conservatism and fiscal policy. *Journal of Monetary Economics 55*(8), 1376–1388.

Barro, R. J. (1979). On the determination of the public debt. *Journal of political Economy 87*(5, Part 1), 940–971.

Barro, R. J. and D. B. Gordon (1983). Rules, discretion and reputation in a model of monetary policy. *Journal of monetary economics 12*(1), 101–121.

Basso, H. S. (2009). Delegation, time inconsistency and sustainable equilibrium. *Journal of Economic Dynamics and Control 33*(8), 1617–1629.

Benhabib, J., S. Schmitt-Grohé, and M. Uribe (2001). Monetary policy and multiple equilibria. *American Economic Review 91*(1), 167–186.

Bilbiie, F. O. (2019). Optimal forward guidance. *American Economic Journal: Macroeconomics 11*(4), 310–45.

Camous, A. and D. Matveev (2019). Credibility of monetary policy with fiscal conditionality. *Unpublished manuscript*.

Camous, A. and D. Matveev (2022). The central bank strikes back! credibility of monetary policy under fiscal influence. Technical report, Bank of Canada.

Chari, V. V., L. J. Christiano, and M. Eichenbaum (1998). Expectation traps and discretion. *Journal of economic theory 81*(2), 462–492.

Chari, V. V. and P. J. Kehoe (1990). Sustainable plans. *Journal of political economy 98*(4), 783–802.

Christiano, L., M. Eichenbaum, and S. Rebelo (2011). When is the government spending multiplier large? *Journal of Political Economy 119*(1), 78–121.

Clarida, R., J. Gali, and M. Gertler (1999). The science of monetary policy: a new keynesian perspective. *Journal of economic literature 37*(4), 1661–1707.

Cook, D. and M. B. Devereux (2011). Optimal fiscal policy in a world liquidity trap. *European Economic Review 55*(4), 443–462.

Davig, T. and R. S. Gürkaynak (2015). Is optimal monetary policy always optimal? *International Journal of Central Banking 11*(4), 353–382.

Debortoli, D. and R. Nunes (2013). Lack of commitment and the level of debt. *Journal of the European Economic Association 11*(5), 1053–1078.

Devereux, M. (2010). Fiscal deficits, debt, and monetary policy in a liquidity trap. *Documentos de Trabajo (Banco Central de Chile)* (581), 1.

Dixit, A. and L. Lambertini (2003). Interactions of commitment and discretion in monetary and fiscal policies. *American economic review 93*(5), 1522–1542.

Dong, B. and E. R. Young (2019). Forward guidance and credible monetary policy.

Draghi, M. (2019). Twenty years of the ecb's monetary policy. In *Speech by Mario Draghi, President of the ECB, ECB Forum on Central Banking, Sintra, June 18, 2019*, Volume 18.

Eggertsson, G. B. (2006). The deflation bias and committing to being irresponsible. *Journal of money, credit, and Banking 38*(2), 283–321.

Eggertsson, G. B. (2011a). Fiscal multipliers and policy coordination. *Documentos de Trabajo (Banco Central de Chile)* (628), 1.

Eggertsson, G. B. (2011b). What fiscal policy is effective at zero interest rates? *NBER Macroeconomics Annual 25*(1), 59–112.

Eggertsson, G. B. and M. Woodford (2003). The zero bound on interest rates and optimal monetary policy. *Brookings Papers on Economic Activity* (1), 139–234.

Galí, J. (2008). *Monetary policy, inflation, and the business cycle: an introduction to the new Keynesian framework.* Princeton University Press.

Garın, J., R. Lester, and E. Sims (2018). Intermediate macroeconomics. *This Version 3*(0).

Garriga, A. C. (2016). Central bank independence in the world: A new data set. *International Interactions 42*(5), 849–868.

Gnocchi, S. and L. Lambertini (2016). Monetary commitment and the level of public debt. Technical report, Bank of Canada.

Greenwood, R., S. Hanson, J. S. Rudolph, and L. H. Summers (2014). Government debt management at the zero lower bound.

Ireland, P. N. (1997). Sustainable monetary policies. *Journal of Economic Dynamics and Control 22*(1), 87–108.

Jung, T., Y. Teranishi, and T. Watanabe (2005). Optimal monetary policy at the zero-interest-rate bound. *Journal of Money, credit, and Banking 37*(5), 813–835.

Kirsanova, T., S. J. Stehn, and D. Vines (2005). The interactions between fiscal policy and monetary policy. *Oxford Review of Economic Policy 21*(4), 532–564.

Krugman, P. R., K. M. Dominquez, and K. Rogoff (1998). It's baaack: Japan's slump and the return of the liquidity trap. *Brookings Papers on Economic Activity 1998*(2), 137–205.

Kurozumi, T. (2008). Optimal sustainable monetary policy. *Journal of Monetary Economics 55*(7), 1277–1289.

Kydland, F. E. and E. C. Prescott (1977). Rules rather than discretion: The inconsistency of optimal plans. *Journal of political economy 85*(3), 473–491.

Leeper, E. M. (1991). Equilibria under 'active' and 'passive' monetary and fiscal policies. *Journal of monetary Economics 27*(1), 129–147.

Leeper, E. M., C. Leith, and D. Liu (2021). Optimal time-consistent monetary, fiscal and debt maturity policy. *Journal of Monetary Economics 117*, 600–617.

Ljungqvist, L. and T. J. Sargent (2018). *Recursive macroeconomic theory.*

McKay, A., E. Nakamura, and J. Steinsson (2017). The discounted euler equation: A note. *Economica 84*(336), 820–831.

Nakata, T. (2017). Optimal government spending at the zero lower bound: A non-ricardian analysis. *Review of Economic Dynamics 23*, 150–169.

Nakata, T. (2018). Reputation and liquidity traps. *Review of Economic Dynamics 28*, 252–268.

Nakata, T. and S. Schmidt (2019). Conservatism and liquidity traps. *Journal of Monetary Economics 104*, 37–47.

Nakata, T. and T. Sunakawa (2019). Credible forward guidance. Technical report, Finance and Economics Discussion Series 2019-037. Washington: Board of Governors of the Federal Reserve System, https://doi.org/10.17016/FEDS.2019.037.

Nakov, A. (2008). Optimal and simple monetary policy rules with zero floor on the nominal interest rate. *International Journal of Central Banking 4*(2), 73–127.

Rigon, M. and F. Zanetti (2018). Optimal monetary policy and fiscal policy interaction in a non-ricardian economy. *International Journal of Central Banking 14*(3).

Rogoff, K. (1985). The optimal degree of commitment to an intermediate monetary target. *The quarterly journal of economics 100*(4), 1169–1189.

Sargent, T. J., N. Wallace, et al. (1981). Some unpleasant monetarist arithmetic. *Federal reserve bank of minneapolis quarterly review 5*(3), 1–17.

Smets, F. and M. Trabandt (2018). Government debt, the zero lower bound and monetary policy. *Unpublished Manuscript*.

Sydsæter, K., P. Hammond, A. Seierstad, and A. Strom (2008). *Further mathematics for economic analysis*. Pearson education.

Walsh, C. (2018). Simple sustainable forward guidance at the elb.

Werning, I. (2012). Managing a liquidity trap: Monetary and fiscal policy, manuscript.

Woodford, M. (2003). *Interest and prices: Foundations of a theory of monetary policy*. Princeton university press.

Woodford, M. (2011). Simple analytics of the government expenditure multiplier. *American Economic Journal: Macroeconomics 3*(1), 1–35.

Woodford, M. (2012). Methods of policy accommodation at the interest-rate lower bound.

Woodford, M. and Y. Xie (2022). Fiscal and monetary stabilization policy at the zero lower bound: Consequences of limited foresight. *Journal of Monetary Economics 125*, 18–35.

Appendix A - Scenario without forward guidance

Derivations of Optimality Conditions without forward guidance

The following solutions closely follow Clarida et al. (1999), Jung et al. (2005), Nakata (2018) and Walsh (2018).

Treasury

Claim 9 *In state R, the treasury's problem (13) can be compactly re-expressed as*

$$\max_{\{b_R,\ell_R,\tau_R\}} -\frac{1}{2}\left(\alpha_y^{Tr}\left(y_R\right)^2+\alpha_\tau^{Tr}\left(\tau_R\right)^2\right)$$

subject to

$$\begin{aligned}
&y_R = y_R^e-\left(i_R-\pi_R^e-r_R\right)+\gamma b_R\\
&b_R=(1+r)\,b_Z-\tau_R+\ell_R\\
&\ell_R=0,\ b_R=0,\ 0\leq b_R\leq\bar{b},\ \ell_R\geq 0,\ \tau_R\geq 0
\end{aligned}$$

where $\{i_R,b_Z,y_R^e,\pi_R^e,r_R,r\}$ *are given.*

Proof of Claim 9.

The general problem for the treasury is

$$\max_{\{\tau_t,b_t,\ell_t\}_t^\infty} -\frac{1}{2}E_t\sum_{k=0}^{\infty}\left(\beta^{Tr}\right)^k\left[\alpha_y^{Tr}y_{t+k}^2+\alpha_\tau^{Tr}\tau_{t+k}^2\right]$$

subject to

$$\begin{aligned}
y_t &= E_t y_{t+1}-\left(i_t-E_t y_{t+1}-r_t\right)+\gamma b_t, && \forall t;\\
b_t &= (1+r)\,b_{t-1}-\tau_t+\ell_t, && \forall t;\\
&\quad 0\leq b_t\leq\bar{b}<\infty, && \forall t;\\
&\quad \ell_t\geq 0, && \forall t;
\end{aligned}$$

and also taking initial debt level b_{t-1}, current central bank's choice i_t and future paths $\{y_{t+k},\ \pi_{t+k},\ i_{t+k},\ \tau_{t+k},\ b_{t+k},\ \ell_{t+k}\}_{k\geq 1}$ as given.

Note first that in state R, the period-t problem of the treasury can be written as

$$\max_{\{b_t,\ell_t,\tau_t\}} -\frac{1}{2}\left(\alpha_y^{Tr}\left(y_R\right)^2+\alpha_\tau^{Tr}\left(\tau_R\right)^2\right)+\Psi_t$$

subject to

$$\begin{aligned}
&y_R = y_R^e-\left(i_R-\pi_R^e-r_R\right)+\gamma b_R\\
&b_R=(1+r)\,b_Z-\tau_R+\ell_R\\
&0\leq b_R\leq\bar{b},\ \text{and}\ \ell_R\geq 0
\end{aligned}$$

and

$$\Psi_t\equiv-\frac{1}{2}E_t\sum_{k=1}^{\infty}\left(\beta^{Tr}\right)^k\left(\alpha_y^{Tr}\left(y_{t+k}\right)^2+\alpha_\tau^{Tr}\left(\tau_{t+k}\right)^2\right)$$

and also taking as given initial debt level b_Z, the central bank's choice i_Z, and variables $\pi_t^e \equiv E_t \pi_{t+1}$ and $y_t^e \equiv E_t y_{t+1}$ representing PS's expectations. For the expectation term in Ψ_t, note first that when the state is R in period t, the state of the world in $t+1$ will be a liquidity trap state Z with probability p, or the absorbing steady state S with probability $(1-p)$. However, note that in the R state of the world, the treasury makes no actual choice due to our assumptions. Hence, we can compute its losses in state R by just dropping term Ψ_t and combining the previous constraints while we make use of our assumptions that the treasury (i) sets $\ell_R = 0$, and (ii) retires past debt with taxation, $b_R = 0$. The previous problem in state R therefore becomes (again, abstracting from Ψ_t since it is independent of today's choices),

$$\max_{\{b_R, \ell_R, \tau_R\}} -\frac{1}{2}\left(\alpha_y^{Tr}(y_R)^2 + \alpha_\tau^{Tr}(\tau_R)^2\right)$$

subject to

$$\begin{aligned}
&y_R = y_R^e - (i_R - \pi_R^e - r_R) + \gamma b_R \\
&b_R = (1+r)\, b_Z - \tau_R + \ell_R \\
&\ell_R = 0,\ b_R = 0,\ 0 \leq b_R \leq \bar{b},\ \text{and } \ell_R \geq 0
\end{aligned}$$

where $\{i_R, b_Z, y_R^e, \pi_R^e, r_R, r\}$ are given. ■

Claim 10 *The solution to the treasury's problem (13) in state R yields*

$$l_R^{Tr} \equiv -\frac{1}{2}\left(\alpha_y^{Tr}\left(y_R^e - i_R + \pi_R^e + r_R\right)^2 + \alpha_\tau^{Tr}\left(\left(1+r_R\right)b_Z\right)^2\right)$$

Proof of Claim 10. Be the problem given by Claim 9

$$\max_{\{b_R, \ell_R, \tau_R\}} -\frac{1}{2}\left(\alpha_y^{Tr}(y_R)^2 + \alpha_\tau^{Tr}(\tau_R)^2\right)$$

subject to

$$\begin{aligned}
&y_R = y_R^e - (i_R - \pi_R^e - r_R) + \gamma b_R \\
&b_R = (1+r)\, b_Z - \tau_R + \ell_R \\
&\ell_R = 0,\ b_R = 0,\ 0 \leq b_R \leq \bar{b},\ \ell_R \geq 0,\ \tau_R \geq 0
\end{aligned}$$

where $\{i_R, b_Z, y_R^e, \pi_R^e, r_R, r_S\}$are given.

Plugging $b_R = 0$, $\ell_R = 0$ and given b_Z and r_R in the treasury's BC yields

$$\tau_R = (1+r)\, b_Z \tag{53}$$

Now, plugging $b_R = 0$ and (i_R, r_R) in the DIS renders

$$y_R = y_R^e - i_R + \pi_R^e + r_R, \tag{54}$$

meaning that $\{y_R\}$ is given to the treasury from the point of view of a problem at period R. Therefore, using Eqs. (53) and (54) in the loss function in periods where state is R ($l_t^{Tr}(s_t = R) \equiv l_R^{Tr}$) yields

$$l_R^{Tr} \equiv -\frac{1}{2}\left(\alpha_y^{Tr}\left(y_R^e - i_R + \pi_R^e + r_R\right)^2 + \alpha_\tau^{Tr}\left((1+r)\, b_Z\right)^2\right). \tag{55}$$

■

Claim 11 *The treasury's problem (??) can be compactly re-expressed in state Z as*

$$\max_{\{b_Z\}} -\frac{1}{2}\alpha_y^{Tr}\left(y_Z^e - (i_Z - \pi_Z^e - r_Z) + \gamma b_Z\right)^2 + \beta^{Tr} E_t l_{t+1}^{Tr}$$

subject to

$$\begin{aligned}
&\pi_Z = \kappa y_Z + \beta^{PS}\pi_Z^e \\
&0 \leq b_Z \leq \bar{b}, \\
&r_Z \text{ given} \\
&l_{t+1}^{Tr} \equiv l_{t+1}^{Tr}(y_{t+1}, \tau_{t+1}) \equiv l_{t+1}^{Tr}\left(i_{t+1}, b_t, b_{t+1}, \ell_{t+1}, y_{t+1}^e, \pi_{t+1}^e, r_{t+1}\right)
\end{aligned}$$

and given $\{i_Z, y_Z^e, \pi_Z^e\}$ (for the first term of the objective) and $\left\{i_{t+1}, b_{t+1}, \ell_{t+1}, \tau_{t+1}, y_{t+1}^e, \pi_{t+1}^e, r_{t+1}\right\}$ (for the second term of the objective) – to make the current state explicit, state-Z variables are only written as $x_t = x_Z$.

Proof of Claim 11.

Note first that in state Z, the period t problem of the treasury can be written as

$$\max_{\{b_t, \ell_t, \tau_t\}} -\frac{1}{2}\left(\alpha_y^{Tr}(y_t)^2 + \alpha_\tau^{Tr}(\tau_t)^2\right) + \beta^{Tr} E_t l_{t+1}^{Tr}(y_{t+1}, \tau_{t+1}) + \Psi_t'$$

subject to

$$\begin{aligned}
&y_t = y_t^e - (i_t - \pi_t^e - r_t) + \gamma b_t \\
&b_t = (1+r)\, b_{t-1} - \tau_t + \ell_t \\
&0 \leq b_t \leq \bar{b},\ \ell_t \geq 0,\ \tau_t \geq 0 \\
&\{b_{t-1}, r_t, i_t, y_t^e, \pi_t^e\} \text{ given}
\end{aligned}$$

and

$$l_{t+1}^{Tr} \equiv l_{t+1}^{Tr}(y_{t+1}, \tau_{t+1}) \equiv l_{t+1}^{Tr}\left(i_{t+1}, b_t, b_{t+1}, \ell_{t+1}, y_{t+1}^e, \pi_{t+1}^e, r_{t+1}, r\right),$$

with $\left\{i_{t+1}, b_{t+1}, \ell_{t+1}, \tau_{t+1}, y_{t+1}^e, \pi_{t+1}^e, r_{t+1}\right\}$ given (and where we made explicit the second term of the summation since it depends on b_{t+1}), and also subject to

$$\Psi_t' \equiv -\frac{1}{2} E_t \sum_{k=2}^{\infty} \beta^j l_{t+k}^{Tr}, \text{ with } l_{t+k}^{Tr} \equiv \alpha_y^{Tr}(y_{t+k})^2 + \alpha_\tau^{Tr}(\tau_{t+k})^2.$$

where again $\{i_{t+k}, b_{t+k}, \ell_{t+k}, \tau_{t+k}, r_{t+k}\}$ for $k \geq 2$. Plugging $y_t = y_t^e - (i_t - \pi_t^e - r_t) + \gamma b_t$ and $\tau_t = 0$ (by assumption of period $s_t = Z$) in the objective, linking the remaining constrains (together with the fact that $b_{t-1} = 0$ in Z states), and dropping the Ψ_t' term (the treasury's choice at t will not affect any variable at $t+2$), then if the current state is Z (and we rename period-t variables using the state subscript), then the treasury's problem becomes

$$\max_{\{b_Z\}} -\frac{1}{2}\alpha_y^{Tr}\left(y_Z^e - (i_Z - \pi_Z^e - r_Z) + \gamma b_Z\right)^2 + \beta^{Tr} E_t l_{t+1}^{Tr}$$

subject to

$$\begin{aligned}
&\pi_Z = \kappa y_Z + \beta^{PS}\pi_Z^e \\
&0 \leq b_Z \leq \bar{b}, \\
&l_{t+1}^{Tr} \equiv l_{t+1}^{Tr}(y_{t+1}, \tau_{t+1}) \equiv l_{t+1}^{Tr}\left(i_{t+1}, b_t, b_{t+1}, \ell_{t+1}, y_{t+1}^e, \pi_{t+1}^e, r_{t+1}\right)
\end{aligned}$$

and given r_Z and $\{i_Z, y_Z^e, \pi_Z^e\}$ (for the first term of the objective) and $\left\{i_{t+1}, b_{t+1}, \ell_{t+1}, \tau_{t+1}, y_{t+1}^e, \pi_{t+1}^e, r_{t+1}\right\}$

(for the second term of the objective). ■

Lemma 12 *Without forward guidance, the optimal level of debt that solves the treasury's problem is,*

$$b_Z = \frac{\alpha_y^{Tr}\gamma}{\alpha_y^{Tr}\gamma^2 + \beta^{Tr}\alpha_\tau^{Tr}(1+r_R)^2}\left(i_Z - r_Z - y_Z^e - \pi_Z^e\right)$$

Proof of Lemma 12. I build this problem using a backward induction logic. In period $t+1$ the state is R, and therefore after the natural interest rate is realized at r_R, the payoff of a treasury in state R (Claim 10) is given by

$$l_R^{Tr} \equiv -\frac{1}{2}\left(\alpha_y^{Tr}\left(y_R^e - i_R + \pi_R^e + r_R\right)^2 + \alpha_\tau^{Tr}\left(\left(1+r_R\right)b_Z\right)^2\right).$$

Variable b_Z is going to be determined by the fiscal authority from state Z (i_R is determined by the monetary authority making decisions in state R, and the same applies for PS's decisions over (y_R^e, π_R^e)).

In period t, the state is Z and the fiscal authority making decisions at Z solves the problem given by Claim 11,

$$\max_{\{b_Z,\ell_Z,\tau_Z\}} -\frac{1}{2}\alpha_y^{Tr}\left(y_Z^e - \left(i_Z - \pi_Z^e - r_Z\right) + \gamma b_Z\right)^2 + \beta^{Tr}E_t l_{t+1}^{Tr}$$

subject to r_Z given,

$$\begin{aligned}&0 \leq b_Z \leq \bar{b},\\ &l_{t+1}^{Tr} \equiv l_{t+1}^{Tr}\left(y_{t+1},\tau_{t+1}\right) \equiv l_{t+1}^{Tr}\left(i_{t+1}, b_t, b_{t+1}, \ell_{t+1}, y_{t+1}^e, \pi_{t+1}^e, r_{t+1}\right)\end{aligned}$$

The fiscal authority is discretionary, but it faces a limited commitment problem in the sense that it honours debt issued in Z states. Furthermore, since our Markov structure for the shock implies that the fiscal authority expects a state R with certainty, the expectation term from $\beta^{Tr}E_t l_{t+1}^{Tr}$ is equal to $\beta^{Tr}l_R^{Tr}$ – with the variables inside l_R^{Tr} at state R. Plugging in this continuation value the expression for l_R^{Tr} from above, the discretionary treasury at state Z solves a two-period problem of the form

$$\max_{\{b_Z,\ell_Z,\tau_Z\}} -\frac{1}{2}\alpha_y^{Tr}\left(y_Z^e - i_Z + \pi_Z^e + r_Z + \gamma b_Z\right)^2 - \beta^{Tr}\frac{1}{2}\left(\alpha_y^{Tr}\left(y_R^e - i_R + \pi_R^e + r_R\right)^2 + \alpha_\tau^{Tr}\left(\left(1+r_R\right)b_Z\right)^2\right)$$

subject to

$$0 \leq b_Z \leq \bar{b},$$

and $\{i_z, r_Z, y_Z^e, \pi_Z^e, i_R, r_R, y_R^e, \pi_R^e\}$ given. The Lagrangean of the treasury's problem at Z is

$$\begin{aligned}\mathcal{L} \;\; = \;\; & -\frac{1}{2}\alpha_y^{Tr}\left(y_Z^e - i_Z + \pi_Z^e + r_Z + \gamma b_Z\right)^2 - \beta^{Tr}\frac{1}{2}\left(\alpha_y^{Tr}\left(y_R^e - i_R + \pi_R^e + r_R\right)^2 + \alpha_\tau^{Tr}\left(\left(1+r_R\right)b_Z\right)^2\right)\\ & -\lambda_1\left(-b_Z\right) - \lambda_2\left(b^Z - \bar{b}\right)\end{aligned}$$

FOC w.r.t. b_Z yields

$$-\frac{1}{2}2\alpha_y\left(y_Z^e - i_Z + \pi_Z^e + r_Z + \gamma b_Z\right)\left(\frac{dy_Z^e}{db_Z} + \frac{d\pi_Z^e}{db_Z} + \gamma\right) - \beta^{Tr}\frac{1}{2}\alpha_\tau^{Tr}2\left(\left(1+r_R\right)b_Z\right)\left(1+r_R\right) - \lambda_1(-1) - \lambda_2(1) = 0$$

Throughout the analysis we look for an interior solution for fiscal policy. Hence, ($\lambda_1 = 0$ and $\lambda_2 = 0$) – the latter condition also implies nonnegative lump-sum transfers. Rearranging terms yields

$$-\alpha_y^{Tr}\left(y_Z^e - i_Z + \pi_Z^e + r_Z + \gamma b_Z\right)\left(\frac{dy_Z^e}{db_Z} + \frac{d\pi_Z^e}{db_Z} + \gamma\right) - \beta^{Tr}\alpha_\tau^{Tr}\left(1+r_R\right)^2 b_Z = 0$$

Terms dy_Z^e/db_Z and $d\pi_Z^e/db_Z$ capture the fact that the treasury is choosing debt (which is a state variable for the next period) which can potentially affect future output gap and inflation – and, therefore, have an impact in state Z via current PS expectations. Several works in the literature treat these expressions as partial derivatives of some function which is usually assumed to be differentiable – see, for example, Debortoli and Nunes (2013), Gnocchi and Lambertini (2016), and Leeper et al. (2021). To compute the equilibrium, I will follow a similar approach and also conjecture that y_Z^e and π_Z^e will depend on next period state variables (b_t, r_{t+1}); namely, $y_t^e \equiv E_t Y\left(b_t, r_{t+1}\right)$ and $\pi_t^e \equiv E_t \Pi\left(b_t, r_{t+1}\right)$. Function $Y\left(b_t, r_{t+1}\right)$ is an auxiliary function denoting the expected output gap in the next period as a function of the debt level chosen in period t (and outstanding, in period $t+1$) for every exogenous state r_{t+1} – the same idea applies to $\Pi\left(b_t, r_{t+1}\right)$. Thus, rewriting terms in the the FOC yield

$$\begin{aligned} 0 \;=\;& -\alpha_y^{Tr}\left(E_tY\left(b_Z, r_{t+1}\right) - i_Z + E_t\Pi\left(b_Z, r_{t+1}\right) + r_Z + \gamma b_Z\right)\left(\frac{dE_tY\left(b_Z, r_{t+1}\right)}{db_Z} + \frac{dE_t\Pi\left(b_Z, r_{t+1}\right)}{db_Z} + \gamma\right) \\ & -\beta^{Tr}\alpha_\tau^{Tr}\left(1+r_R\right)^2 b_Z \end{aligned}$$

Applying the Interchange of Integration and Differentiation Theorem we can take the derivatives of the expression inside the expectation operators,

$$\begin{aligned} 0 \;=\;& -\alpha_y^{Tr}\left(E_tY\left(b_Z, r_{t+1}\right) - i_Z + E_t\Pi\left(b_Z, r_{t+1}\right) + r_Z + \gamma b_Z\right)\left(E_t\frac{dY\left(b_Z, r_{t+1}\right)}{db_Z} + E_t\frac{d\Pi\left(b_Z, r_{t+1}\right)}{db_Z} + \gamma\right) \\ & -\beta^{Tr}\alpha_\tau^{Tr}\left(1+r_R\right)^2 b_Z \end{aligned}$$

Although these auxiliary functions are very generic, in equilibrium a clean closed-form solution to the model can still be obtained given our assumptions when the economy is in state R. Specifically, recall that for whatever debt level b_Z inherited in periods where the state is R, we argued that (i) the fiscal authority will retire outstanding debt with taxation (i.e., $\tau_R \geq 0$) , and (ii) the treasury does not issue any additional debt in state R ($b_R = 0$) or give any extra lump-sum transfers ($\ell_R = 0$). The combination of (i) and (ii) implies that any existing debt b_Z inherited in the R state is absorbed via taxes, and not via debt financing. But, furthermore, since b_R enters in the private sector's DIS (but it is fixed at 0), then any outstanding level of debt b_Z does not affect PS's decisions via b_R since fiscal policy only retires debt in Recovery times with taxes. As a result of this, $\frac{dY(b_Z, r_{t+1})}{db_Z} = 0$ and $\frac{d\Pi(b_Z, r_{t+1})}{db_Z} = 0$, and plugging this in the last F.O.C. expression yields

$$\left(\alpha_y^{Tr}\gamma^2 + \beta^{Tr}\alpha_\tau^{Tr}\left(1+r_R\right)^2\right) b_Z = -\alpha_y^{Tr}\gamma E_tY\left(r_{t+1}\right) + \alpha_y^{Tr}\gamma i_Z - \alpha_y^{Tr}\gamma E_t\Pi\left(r_{t+1}\right) - \alpha_y^{Tr}\gamma r_Z,$$

where b_Z yields

$$b_Z = \Phi\left(i_Z - y_Z^e - \pi_Z^e - r_Z\right)$$

with $\Phi \equiv \alpha_y^{Tr}\gamma/\left(\alpha_y^{Tr}\gamma^2 + \beta^{Tr}\alpha_\tau^{Tr}\left(1+r_R\right)\right)$ – and recall that $(1+r_S) = (1+r_R) = (1+r)$. ■

Central Bank

The solution to the central bank's problem for each state is conducted in two steps. First, Lemma 13 will solve the optimization problem using a generic fiscal policy rule with given coefficients. Then, Lemma 14 will cast the results from the previous Lemma using the fiscal policy obtained from Lemma 11. All these results are calculated using a version of the ZLB given by $i_t \geq \iota$ – but, in the paper, ι is set at 0.

Lemma 13 *During the Recovery state, the discretionary Central bank sets the 'targeting rule'*

$$y_R + \frac{\alpha_\pi^{CB}\kappa}{\alpha_y^{CB}}\pi_R = 0,$$

with associated inflation, output gap, and nominal interest rate given by

$$\pi_R = \frac{\alpha_y^{CB}}{\alpha_y^{CB}+\alpha_\pi^{CB}\kappa^2}\beta^{PS}\pi_R^e, \qquad y_R = -\frac{\alpha_\pi^{CB}\kappa}{\alpha_y^{CB}+\alpha_\pi^{CB}\kappa^2}\beta^{PS}\pi_R^e,$$

and

$$i_R = y_R^e + \left(1 + \frac{\alpha_\pi^{CB}\kappa\beta^{PS}}{\alpha_y^{CB}+\alpha_\pi^{CB}\kappa^2}\right)\pi_R^e + r_R + \gamma b_R \geq 0.$$

During the ZLB state, the CB sets the 'targeting rule'

$$\left(1-\gamma\Phi_Z^i\right)\left(\alpha_y^{CB}+\alpha_\pi^{CB}\kappa^2\right)\left(y_t + \frac{\alpha_\pi^{CB}\kappa\beta^{PS}}{\alpha_y^{CB}+\alpha_\pi^{CB}\kappa^2}\pi_t^e\right) < 0$$

with associated inflation, output gap, and nominal interest rate given by

$$\pi_Z = \left(\left(1+\gamma\Phi_Z^y\right)\kappa\right)y_Z^e + \left(\left(1+\gamma\Phi_Z^\pi\right)\kappa + \beta^{PS}\right)\pi_Z^e - \left(\kappa\left(1-\gamma\Phi_Z^i\right)\right)\iota + \left(\left(1+\gamma\Phi_Z^r\right)\kappa\right)r_Z$$

$$y_Z = \left(1+\gamma\Phi_Z^y\right)y_Z^e + \left(1+\gamma\Phi_Z^\pi\right)\pi_Z^e - \left(1-\gamma\Phi_Z^i\right)\iota + \left(1+\gamma\Phi_Z^r\right)r_Z$$

and $i_Z = 0$, with implicit i_Z satisfying

$$(i_Z =)\,\iota > \frac{1+\gamma\Phi_Z^y}{1-\gamma\Phi_Z^i}y_Z^e + \frac{1+\gamma\Phi_Z^\pi + A}{1-\gamma\Phi_Z^i}\pi_Z^e + \frac{1+\gamma\Phi_Z^r}{1-\gamma\Phi_Z^i}r_Z$$

which is equivalent to the satisfaction of the following condition:

$$\iota\left(\frac{1-\gamma\Phi_Z^i}{1+\gamma\Phi_Z^r}\right) - \frac{1+\gamma\Phi_Z^y}{1-\gamma\Phi_Z^i}\left(\frac{1-\gamma\Phi_Z^i}{1+\gamma\Phi_Z^r}\right)y_Z^e - \frac{1+\gamma\Phi_Z^\pi + A}{1-\gamma\Phi_Z^i}\left(\frac{1-\gamma\Phi_Z^i}{1+\gamma\Phi_Z^r}\right)\pi_Z^e > r_Z.$$

Proof of Lemma 13.

Variables $\pi_t^e \equiv E_t\pi_{t+1}$ and $y_t^e \equiv E_t y_{t+1}$ are fixed. Consider then a treasury's choice b_t given by

$$b_t = \Phi_Z^r r_t + \Phi_Z^i i_t + \Phi_Z^y y_t^e + \Phi_Z^\pi \pi_t^e$$

when the ZLB binds, and $b_t = 0$ away from the ZLB. Variable r_t is the exogenous shock. First, note that the ZLB can be re-written as

$$i_t \geq \iota \Rightarrow -i_t \leq -\iota \Rightarrow -i_t + \iota \leq 0$$

At the ZLB, expression for b_t in the DIS yields

$$y_t = y_t^e - (i_t - \pi_t^e - r_t) + \gamma \left(\Phi_Z^r r_t + \Phi_Z^i i_t + \Phi_Z^y y_t^e + \Phi_Z^\pi \pi_t^e \right)$$

Following an exposition similar to Jung et al. (2005), I calculate the two different cases for states Z and R. The Lagrangian at Z yields

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\left(\alpha_y^{CB} (y_t)^2 + \alpha_\pi^{CB} (\pi_t)^2 \right) - \lambda_t^1 \left[\pi_t - \kappa y_t - \beta^{PS} \pi_t^e \right] \\ &\quad -\lambda_t^2 \left[y_t - y_t^e + (i_t - \pi_t^e - r_t) - \gamma \left(\Phi_Z^r r_t + \Phi_Z^i i_t + \Phi_Z^y y_t^e + \Phi_Z^\pi \pi_t^e \right) \right] - \lambda_t^3 \left[-i_t + \iota \right] \end{aligned}$$

F.O.C.'s with respect to π_t, y_t and i_t render

$$\frac{\partial \mathcal{L}}{\partial y_t} = 0 \Rightarrow y_t = \frac{\kappa}{\alpha_y^{CB}} \lambda_t^1 - \frac{1}{\alpha_y^{CB}} \lambda_t^2 \tag{56}$$

$$\frac{\partial \mathcal{L}}{\partial \pi_t} = 0 \Rightarrow \lambda_t^1 = -\alpha_\pi^{CB} \pi_t \tag{57}$$

$$\frac{\partial \mathcal{L}}{\partial i_t} = 0 \Rightarrow \lambda_t^2 = \frac{\lambda_t^3}{1 - \gamma \Phi_Z^i} \tag{58}$$

For the equality constraints, it must be that

$$\frac{\partial \mathcal{L}}{\partial \lambda_t^1} = 0 \Rightarrow \pi_t = \kappa y_t + \beta^{PS} \pi_t^e, \tag{59}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t^2} = 0 \Rightarrow y_t = y_t^e - (i_t - \pi_t^e - r_t) + \gamma \left(\Phi_Z^r r_t + \Phi_Z^i i_t + \Phi_Z^y y_t^e + \Phi_Z^\pi \pi_t^e \right). \tag{60}$$

For the inequality constraint,

$$\lambda_t^3 (-i_t + \iota) = 0 \tag{61}$$

with

$$\lambda_t^3 \geq 0 \tag{62}$$

and

$$i_t \geq \iota \tag{63}$$

Equations (57), (56) and (58) renders

$$y_t = -\frac{\alpha_\pi^{CB} \kappa}{\alpha_y^{CB}} \pi_t - \frac{1}{\alpha_y^{CB} \left(1 - \gamma \Phi_Z^i\right)} \lambda_t^3 \tag{64}$$

In the Z state, we look at the case where $\lambda_t^3 > 0$ and the ZLB constraint binds. By the complementary slackness condition (Eq. (61)), it must be that

$$i_t = \iota \tag{65}$$

Then, Eq. (64) becomes

$$\left(1 - \gamma \Phi_Z^i\right) \left(\alpha_y^{CB} y_t + \alpha_\pi^{CB} \kappa \pi_t \right) = -\underbrace{\lambda_t^3}_{>0} < 0 \tag{66}$$

and using the NKPC and DIS (Eqs. (59) and (60)), we arrive at

$$\left(1-\gamma\Phi_Z^i\right)\left(\alpha_y^{CB}+\alpha_\pi^{CB}\kappa^2\right)\left(y_t+\frac{\alpha_\pi^{CB}\kappa\beta^{PS}}{\alpha_y^{CB}+\alpha_\pi^{CB}\kappa^2}\pi_t^e\right)<0$$

Disregarding $\left(\alpha_y^{CB}+\alpha_\pi^{CB}\kappa^2\right)>0$, denoting $A\equiv\frac{\alpha_\pi^{CB}\kappa\beta^{PS}}{\alpha_y^{CB}+\alpha_\pi^{CB}\kappa^2}$, and assuming that $\left(1-\gamma\Phi_Z^i\right)>0$ (we will later in the endogenous fiscal policy case that this can be the case), then

$$\frac{1+\gamma\Phi_Z^y}{1-\gamma\Phi_Z^i}y_Z^e+\frac{1+\gamma\Phi_Z^r}{1-\gamma\Phi_Z^i}r_z+\frac{1+\gamma\Phi_Z^\pi+A}{1-\gamma\Phi_Z^i}\pi_Z^e<i_Z$$

and Eq. (65) implies that

$$(i_Z=)\,\iota>\frac{1+\gamma\Phi_Z^y}{1-\gamma\Phi_Z^i}y_Z^e+\frac{1+\gamma\Phi_Z^\pi+A}{1-\gamma\Phi_Z^i}\pi_Z^e+\frac{1+\gamma\Phi_Z^r}{1-\gamma\Phi_Z^i}r_z \tag{67}$$

So, the ZLB binds for some values of r_Z: The conditions under which r_t is negative enough to make $i_t=\iota$ are:

$$\iota>\frac{1+\gamma\Phi_Z^y}{1-\gamma\Phi_Z^i}y_Z^e+\frac{1+\gamma\Phi_Z^\pi+A}{1-\gamma\Phi_Z^i}\pi_Z^e+\frac{1+\gamma\Phi_Z^r}{1-\gamma\Phi_Z^i}r_z\Rightarrow\iota-\frac{1+\gamma\Phi_Z^y}{1-\gamma\Phi_Z^i}y_Z^e-\frac{1+\gamma\Phi_Z^\pi+A}{1-\gamma\Phi_Z^i}\pi_Z^e>\frac{1+\gamma\Phi_Z^r}{1-\gamma\Phi_Z^i}r_Z$$

Since$\frac{1+\gamma\Phi_Z^r}{1-\gamma\Phi_Z^i}>0$, and if $\gamma\Phi_Z^i\in(0,1)$, then

$$\iota\left(\frac{1-\gamma\Phi_Z^i}{1+\gamma\Phi_Z^r}\right)-\frac{1+\gamma\Phi_Z^y}{1-\gamma\Phi_Z^i}\left(\frac{1-\gamma\Phi_Z^i}{1+\gamma\Phi_Z^r}\right)y_Z^e-\frac{1+\gamma\Phi_Z^\pi+A}{1-\gamma\Phi_Z^i}\left(\frac{1-\gamma\Phi_Z^i}{1+\gamma\Phi_Z^r}\right)\pi_Z^e>r_Z \tag{68}$$

(Note that if $\Phi_Z^i=\Phi_Z^r=\Phi_Z^y=\Phi_Z^\pi$ then the previous expression becomes $\iota\left(\frac{1-\gamma\Phi_Z^i}{1+\gamma\Phi_Z^r}\right)-y_Z^e-\frac{1+\gamma\Phi_Z^\pi+A}{1+\gamma\Phi_Z^r}\pi_Z^e>r_Z$. We will show that this can be the case when we use the result from the treasury optimization problem.) Finally, plugging $i_t=\iota$ and optimal choice of Tr, $b_Z=\Phi_Z^r r_Z+\Phi_Z^i\iota+\Phi_Z^y y_Z^e+\Phi_Z^\pi\pi_Z^e$ in the NKPC and DIS, and using subscripts to denote the state yields

$$\pi_Z=\left(\left(1+\gamma\Phi_Z^y\right)\kappa\right)y_Z^e+\left(\left(1+\gamma\Phi_Z^\pi\right)\kappa+\beta^{PS}\right)\pi_Z^e-\left(\kappa\left(1-\gamma\Phi_Z^i\right)\right)\iota+\left(\left(1+\gamma\Phi_Z^r\right)\kappa\right)r_Z \tag{69}$$

$$y_Z=\left(1+\gamma\Phi_Z^y\right)y_Z^e+\left(1+\gamma\Phi_Z^\pi\right)\pi_Z^e-\left(1-\gamma\Phi_Z^i\right)\iota+\left(1+\gamma\Phi_Z^r\right)r_Z \tag{70}$$

$$i_Z=0 \tag{71}$$

and implicit i_Z satisfying

$$(i_Z=)\,\iota>\frac{1+\gamma\Phi_Z^y}{1-\gamma\Phi_Z^i}y_Z^e+\frac{1+\gamma\Phi_Z^\pi+A}{1-\gamma\Phi_Z^i}\pi_Z^e+\frac{1+\gamma\Phi_Z^r}{1-\gamma\Phi_Z^i}r_z \tag{72}$$

which is equivalent to the satisfaction of the following condition:

$$\iota\left(\frac{1-\gamma\Phi_Z^i}{1+\gamma\Phi_Z^r}\right)-\frac{1+\gamma\Phi_Z^y}{1-\gamma\Phi_Z^i}\left(\frac{1-\gamma\Phi_Z^i}{1+\gamma\Phi_Z^r}\right)y_Z^e-\frac{1+\gamma\Phi_Z^\pi+A}{1-\gamma\Phi_Z^i}\left(\frac{1-\gamma\Phi_Z^i}{1+\gamma\Phi_Z^r}\right)\pi_Z^e>r_Z \tag{73}$$

This finalizes our relevant results for state Z.

At R, the Lagrangian for the central bank is

$$\mathcal{L} = -\frac{1}{2}\left(\alpha_y^{CB}(y_t)^2 + \alpha_\pi^{CB}(\pi_t)^2\right) - \lambda_t^1\left[\pi_t - \kappa y_t - \beta^{PS}\pi_t^e\right] - \lambda_t^2[y_t - y_t^e + (i_t - \pi_t^e - r_t) - \gamma\underbrace{b_t}_{=0}] - \lambda_t^3\left[-i_t + \iota\right]$$

F.O.C.'s with respect to π_t, y_t and i_t render

$$\frac{\partial\mathcal{L}}{\partial y_t} = 0 \Rightarrow y_t = \frac{\kappa}{\alpha_y^{CB}}\lambda_t^1 - \frac{1}{\alpha_y^{CB}}\lambda_t^2 \tag{74}$$

$$\frac{\partial\mathcal{L}}{\partial \pi_t} = 0 \Rightarrow \lambda_t^1 = -\alpha_\pi^{CB}\pi_t \tag{75}$$

$$\frac{\partial\mathcal{L}}{\partial i_t} = 0 \Rightarrow \lambda_t^2 = \lambda_t^3 \tag{76}$$

For the equality constraints, it must be that

$$\frac{\partial\mathcal{L}}{\partial \lambda_t^1} = 0 \Rightarrow \pi_t = \kappa y_t + \beta^{PS}\pi_t^e, \tag{77}$$

$$\frac{\partial\mathcal{L}}{\partial \lambda_t^2} = 0 \Rightarrow y_t = y_t^e - (i_t - \pi_t^e - r_t) + \gamma b_t. \tag{78}$$

For the inequality constraint,

$$\lambda_t^3\left(-i_t + \iota\right) = 0 \tag{79}$$

with

$$\lambda_t^3 \geq 0 \tag{80}$$

and

$$i_t \geq \iota \tag{81}$$

Equations (75) in (74) (and after plugging (76)) render

$$y_t = -\frac{\alpha_\pi^{CB}\kappa}{\alpha_y^{CB}}\pi_t - \frac{1}{\alpha_y^{CB}}\lambda_t^3 \tag{82}$$

In the R state, the ZLB constraint is not binding. Hence, $\lambda_t^3 = 0$. Therefore, combining equations following the same steps as before yield the solution in the R state defined by Eq. (**??**) (the output targeting rule),

$$y_R = -\frac{\alpha_\pi^{CB}\kappa}{\alpha_y^{CB}}\pi_R \tag{83}$$

and Eqs. (84), (85), (86)

$$\pi_R = \frac{\alpha_y^{CB}}{\alpha_y^{CB} + \alpha_\pi^{CB}\kappa^2}\beta^{PS}\pi_R^e \tag{84}$$

$$y_R = -\frac{\alpha_\pi^{CB}\kappa}{\alpha_y^{CB} + \alpha_\pi^{CB}\kappa^2}\beta^{PS}\pi_R^e \tag{85}$$

$$i_R = y_R^e + \left(1 + \frac{\alpha_\pi^{CB}\kappa\beta^{PS}}{\alpha_y^{CB} + \alpha_\pi^{CB}\kappa^2}\right)\pi_R^e + r_R + \gamma b_R \tag{86}$$

This completes the proof. ■

Lemma 14 *During the Recovery state, the discretionary Central bank sets*

$$y_R + \frac{\alpha_\pi^{CB}\kappa}{\alpha_y^{CB}}\pi_R = 0,$$

with associated inflation, output gap, and nominal interest rate given by

$$\pi_R = \frac{\alpha_y^{CB}}{\alpha_y^{CB}+\alpha_\pi^{CB}\kappa^2}\beta^{PS}\pi_R^e, \qquad y_R = -\frac{\alpha_\pi^{CB}\kappa}{\alpha_y^{CB}+\alpha_\pi^{CB}\kappa^2}\beta^{PS}\pi_R^e,$$

and

$$i_R = y_R^e + \left(1 + \frac{\alpha_\pi^{CB}\kappa\beta^{PS}}{\alpha_y^{CB}+\alpha_\pi^{CB}\kappa^2}\right)\pi_R^e + r_R + \gamma b_R \geq 0.$$

During the ZLB state, the CB sets

$$(1-\gamma\Phi)\left(\alpha_y^{CB}+\alpha_\pi^{CB}\kappa^2\right)\left(y_Z + \frac{\alpha_\pi^{CB}\kappa\beta^{PS}}{\alpha_y^{CB}+\alpha_\pi^{CB}\kappa^2}\pi_Z^e\right) < 0$$

with associated inflation, output gap, and nominal interest rate given by

$$\pi_Z = (1-\gamma\Phi)\kappa y_Z^e + \left((1-\gamma\Phi)\kappa + \beta^{PS}\right)\pi_Z^e - (1-\gamma\Phi)\kappa\iota + (1-\gamma\Phi)\kappa r_Z$$

$$y_Z = (1-\gamma\Phi)\left(y_Z^e + \pi_Z^e - \iota + r_Z\right)$$

and $i_Z = 0$*,and implicit* i_Z *satisfying*

$$(i_Z =)\,\iota > y_Z^e + \left(\frac{1-\gamma\Phi + A}{1-\gamma\Phi}\right)\pi_Z^e + r_Z$$

which is equivalent to the satisfaction of the following condition:

$$\iota - y_Z^e - \left(\frac{1-\gamma\Phi + A}{1-\gamma\Phi}\right)\pi_Z^e > r_Z.$$

Proof of Lemma 14.

For the endogenous fiscal policy case, the treasury's choice b_t in state Z is given by

$$b_t = \Phi\left(i_t - (y_t^e + \pi_t^e) - r_t\right), \tag{87}$$

with $\Phi \equiv \frac{\alpha_y^{Tr}\gamma}{\alpha_y^{Tr}\gamma^2+\beta^{Tr}\alpha_\tau^{Tr}(1+r_R)^2}$. This is equivalent to the exogenous fiscal policy case where $b_t = \Phi_Z^r r_t + \Phi_Z^i i_t + \Phi_Z^y y_t^e + \Phi_Z^\pi \pi_t^e$ and coefficients are given by

$$\Phi_Z^r \equiv -\Phi, \qquad \Phi_Z^y \equiv -\Phi, \qquad \Phi_Z^\pi \equiv -\Phi, \quad \text{and} \quad \Phi_Z^i \equiv \Phi. \tag{88}$$

To prove this, note that if we impose the previous conditions on the exogenous fiscal policy case, then we obtain:

$$b_t = \Phi_Z^r r_t + \Phi_Z^i i_t + \Phi_Z^y y_t^e + \Phi_Z^\pi \pi_t^e \Rightarrow b_t = \Phi\left(i_t - (y_t^e + \pi_t^e) - r_t\right)$$

which is equal to Eq. (87). Therefore, the problems for the CB in states Z and R can be cast without calculating again the optimization problems, but using instead our relations given by expressions (88). Therefore, by only replacing results from the previous Lemma with the equivalences from (88), we obtain the following expressions.

The optimality condition (64) becomes

$$y_t = -\frac{\alpha_\pi^{CB}\kappa}{\alpha_y^{CB}}\pi_t - \frac{1}{\alpha_y^{CB}(1-\gamma\Phi)}\lambda_t^3 \tag{89}$$

Then, imposing the condition on the multiplier, Eq. (89) becomes

$$(1-\gamma\Phi)\left(\alpha_y^{CB} y_t + \alpha_\pi^{CB}\kappa\pi_t\right) < 0 \tag{90}$$

Using the NKPC, Eq. (90) yields

$$(1-\gamma\Phi)\left(\alpha_y^{CB} + \alpha_\pi^{CB}\kappa^2\right)\left(y_t + \frac{\alpha_\pi^{CB}\kappa\beta^{PS}}{\alpha_y^{CB} + \alpha_\pi^{CB}\kappa^2}\pi_t^e\right) < 0$$

and from here it is easy to see that

$$\gamma\Phi = \gamma\left(\frac{\alpha_y^{Tr}\gamma}{\alpha_y^{Tr}\gamma^2 + \beta^{Tr}\alpha_\tau^{Tr}(1+r_R)^2}\right) = \frac{\alpha_y^{Tr}\gamma^2}{\alpha_y^{Tr}\gamma^2 + \beta^{Tr}\alpha_\tau^{Tr}(1+r_R)^2} \in (0,1)$$

implying that $1-\gamma\Phi \in (0,1)$. Thus, we only need that

$$y_t + \frac{\alpha_\pi^{CB}\kappa\beta^{PS}}{\alpha_y^{CB} + \alpha_\pi^{CB}\kappa^2}\pi_t^e < 0.$$

(disregarding $\left(\alpha_y^{CB} + \alpha_\pi^{CB}\kappa^2\right) > 0$). Now, plugging in the DIS, solving for i_t and calling $A \equiv \frac{\alpha_\pi^{CB}\kappa\beta^{PS}}{\alpha_y^{CB}+\alpha_\pi^{CB}\kappa^2}$,

$$y_t^e + r_t + \left(\frac{1-\gamma\Phi + A}{1-\gamma\Phi}\right)\pi_t^e < i_t \tag{91}$$

Imposing condition (88) on expression (91), we can build the equivalent version of Eq. (67),

$$(i_t =)\, \iota > y_t^e + \left(\frac{1-\gamma\Phi + A}{1-\gamma\Phi}\right)\pi_t^e + r_t \tag{92}$$

Lastly, we can build the equivalent version of Eq. (68) by solving Eq. (92) for r_t,

$$\iota - y_t^e - \left(\frac{1-\gamma\Phi + A}{1-\gamma\Phi}\right)\pi_t^e > r_t \tag{93}$$

This expression is what we identify as condition C1 in text (imposing that $\iota = 0$).

Finally, using the DIS and NKPC under the endogenous fiscal choice, and labelling variables with Z, then solution in the liquidity trap state is defined by Eqs.

$$(1-\gamma\Phi)\left(\alpha_y^{CB} + \alpha_\pi^{CB}\kappa^2\right)\left(y_Z + \frac{\alpha_\pi^{CB}\kappa\beta^{PS}}{\alpha_y^{CB} + \alpha_\pi^{CB}\kappa^2}\pi_Z^e\right) < 0 \tag{94}$$

$$\pi_Z = (1-\gamma\Phi)\,\kappa y_Z^e + \left((1-\gamma\Phi)\,\kappa + \beta^{PS}\right)\pi_Z^e - (1-\gamma\Phi)\,\kappa\iota + (1-\gamma\Phi)\,\kappa r_Z \tag{95}$$

$$y_Z = (1-\gamma\Phi)\left(y_Z^e + \pi_Z^e - \iota + r_Z\right) \tag{96}$$

$$i_Z = 0 \tag{97}$$

and implicit i_Z from

$$(i_Z =)\,\iota > y_Z^e + \left(\frac{1-\gamma\Phi+A}{1-\gamma\Phi}\right)\pi_Z^e + r_Z \tag{98}$$

which is equivalent to the satisfaction of the following condition (from Eq. (93)

$$\iota - y_Z^e - \left(\frac{1-\gamma\Phi+A}{1-\gamma\Phi}\right)\pi_Z^e > r_Z \tag{99}$$

This finalizes our relevant results for state Z. Finally, note that away from the ZLB (state R) the problem is the same as before, so it is not necessary to make any adjustments to the expressions previously obtained. ■

Claim 15 *Be T^* a random period where the natural interest rate returns to its steady state, $r_t = r_S$. The economy then exhibits two candidate steady states:*

1. *A zero-inflation steady state characterized by*

$$\pi_S = 0,\ y_S = 0,\ i_S = r_S,\ \tau_S = 0,\ \ell_S = 0,\ b_S = 0,\ \lambda_S^1 = 0,\ \lambda_S^2 = 0,\ \lambda_S^3 = 0$$

(and the PS rationally expects $(\pi_t^e, y_t^e) = (\pi_S, y_S) = (0,0)$),

2. *A steady state with deflationary expectations characterized by*

$$\pi_S = -r_S,\ y_S = -\frac{1-\beta^{PS}}{\kappa}r_S,\ i_S = 0,\ \tau_S = 0,\ \ell_S = 0,\ b_S = 0,$$

$$\lambda_S^1 = \alpha_\pi^{CB} r_S,\ \lambda_S^2 = \left(\alpha_y^{CB}\frac{1-\beta^{PS}}{\kappa} + \alpha_\pi^{CB}\kappa\right)r_S,\ \lambda_S^3 = \left(\alpha_y^{CB}\frac{1-\beta^{PS}}{\kappa} + \alpha_\pi^{CB}\kappa\right)r_S$$

(and the PS rationally expects $(\pi_t^e, y_t^e) = (\pi_S, y_S) = \left(-r_S, -\frac{1-\beta^{PS}}{\kappa}r_S\right) < (0,0)$).

Proof of Claim 15. The following exposition closely follows Jung et al. (2005). Assume that the economy is at $t = T^*$. First recall that we assume that fiscal instruments are perfectly correlated with the shock. Hence, for the treasury this implies that the applied policy when the shock $(\tau_t, \ell_t, b_t) = (0,0,0)$. Likewise, the Central bank's problem yielded the following FOC's

$$\begin{aligned}
\tfrac{\partial\mathcal{L}}{\partial y_t} = 0 \Rightarrow \quad & y_t = \tfrac{\kappa}{\alpha_y^{CB}}\lambda_t^1 - \tfrac{1}{\alpha_y^{CB}}\lambda_t^2 \\
\tfrac{\partial\mathcal{L}}{\partial \pi_t} = 0 \Rightarrow \quad & \lambda_t^1 = -\alpha_\pi^{CB}\pi_t \\
\tfrac{\partial\mathcal{L}}{\partial i_t} = 0 \Rightarrow \quad & \lambda_t^2 = \lambda_t^3 \\
\tfrac{\partial\mathcal{L}}{\partial \lambda_t^1} = 0 \Rightarrow \quad & \pi_t = \kappa y_t + \beta^{PS}\pi_t^e \\
\tfrac{\partial\mathcal{L}}{\partial \lambda_t^2} = 0 \Rightarrow \quad & y_t = y_t^e - (i_t - \pi_t^e - r_t) + \gamma b_{t+1} \\
& \lambda_t^3(-i_t + \iota) = 0,\ \lambda_t^3 \ge 0,\ i_t \ge \iota
\end{aligned} \tag{100}$$

Finally, recall that in the absorbing state S, uncertainty is resolved. Therefore, for the private sector, $\pi_t^e \equiv E_t\pi_{t+1} = \pi_{t+1}$ and $y_t^e \equiv E_t y_{t+1} = y_{t+1}$.

Now, from Lemma **??**, we can combine the first three equations and, after plugging the private sector expectations, we obtain the following system of equations

$$\begin{array}{l} y_S = -\frac{\alpha_\pi^{CB}\kappa}{\alpha_y^{CB}}\pi_S - \frac{1}{\alpha_y^{CB}}\lambda_S^3 \\ \pi_S = \frac{\kappa}{1-\beta^{PS}} y_S \\ i_S = \pi_S + r_S \\ \lambda_S^3 i_S = 0,\ \lambda_S^3 \geq 0,\ i_S \geq 0 \\ \tau_S = 0,\ \ell_S = 0,\ b_S = 0 \end{array}$$

(Steady-state variables denoted as π_S, y_S, τ_S, ℓ_S, b_S, λ_S^1, λ_S^2 and λ_S^3 have already been replaced in the treasury and central bank's policy functions, and private sector expectations. Also note that we present a system where we plugged $b_S = 0$ in DIS, simplified the NKPC, and set $\iota = 0$.)

We can construct and obtain an equilibrium with zero-inflation (which will generate zero-output gap in steady state, and positive nominal interest rates). This equilibrium outcome is obtained when the ZLB condition is not binding: If $\lambda_S^3 = 0$ and $i_S > 0$, condition $\lambda_S^3 i_S = 0$ is satisfied, the first equation is

$$y_S = -\frac{\alpha_\pi^{CB}\kappa}{\alpha_y^{CB}}\pi_S$$

Plugging it in the NKPC, it is easy to see that the equality holds if $\pi_S = 0$. Taking this result to the NKPC yields a zero-inflation steady state characterized by

$$\pi_S = 0,\ y_S = 0,\ i_S = r_S,\ \tau_S = 0,\ \ell_S = 0,\ b_S = 0,\ \lambda_S^1 = 0,\ \lambda_S^2 = 0,\ \lambda_S^3 = 0$$

(and the PS rationally expects $(\pi_t^e, y_t^e) = (\pi_S, y_S) = (0, 0)$).

I now proceed to build an equilibrium with deflationary expectations (also known as "self-fulfilling deflationary spiral", see for example Jung et al. (2005)). Using our system of equations 100, conjecture that $\pi_{t+1} < 0$. In steady state, $\pi_t = \pi_{t+1} = \pi_S$, and since $\pi_{t+1} < 0$, then $\pi_S < 0$. Also, $y_t = y_S$, and from the NKPC it can be rewritten as $y_S = \left(1 - \beta^{PS}\right)\pi_S/\kappa$. Provided that $\pi_S < 0$, then $y_S < 0$, and this implies that the CB's FOC in steady state is $-\left(\alpha_y^{CB} y_S + \alpha_\pi^{CB}\kappa\pi_S\right) = \lambda_S^3$. As $y_S < 0$ and $\pi_S < 0$, this implies that $\lambda_S^3 > 0$, which needs that $i_S = 0$. Hence, the DIS in steady state is $i_S = \pi_S + r_S$ and provided that $i_S = 0$ and $r_S > 0$, then $\pi_S = -r_S < 0$ – this proves the initial conjecture that there exists an equilibrium with deflationary expectations. Finally, replacing in variables y_S, λ_S^3, λ_S^2, and λ_S^1, the steady state with deflationary expectations is characterized by

$$\pi_S = -r_S,\ y_S = -\frac{1-\beta^{PS}}{\kappa}r_S,\ i_S = 0,\ \tau_S = 0,\ \ell_S = 0,\ b_S = 0,$$

$$\lambda_S^1 = \alpha_\pi^{CB} r_S,\ \lambda_S^2 = \left(\alpha_y^{CB}\frac{1-\beta^{PS}}{\kappa} + \alpha_\pi^{CB}\kappa\right) r_S,\ \lambda_S^3 = \left(\alpha_y^{CB}\frac{1-\beta^{PS}}{\kappa} + \alpha_\pi^{CB}\kappa\right) r_S$$

(and the PS rationally expects $(\pi_t^e, y_t^e) = (\pi_S, y_S) = \left(-r_S, -\frac{1-\beta^{PS}}{\kappa}r_S\right) < (0, 0)$). ■

Corollary 16 *The equilibrium payoffs when the economy reverts to its steady state at some date t are* 0.

Proof. The equilibrium payoffs when the economy reverts to its steady state at some date t are calculated as follows. As in Walsh (2018), be l_k^j the value of losses in state k, with $k = \{Z, R, S\}$ and $j = \{CB, Tr\}$. Then, given the zero steady state values of inflation, output gap, and taxes, the central bank's and treasury's per-period losses at the zero-inflation steady state become

$$l_S^{CB} \equiv l_t^{CB}\left(y_t^S, \pi_t^S\right) = 0, \tag{101}$$

and

$$l_S^{Tr} \equiv l_t^{Tr}\left(y_t^S, \tau_t^S\right) = 0. \tag{102}$$

■

Proof of Proposition 4. Imposing Private sector's rational expectations from states Z and R ($(y_Z^e, \pi_Z^e) = (\pi_R, y_R)$ and $(y_R^e, \pi_R^e) = (p\pi_Z, py_Z)$ from Eqs. (37) and (36)) on DIS and NKPC in the ZLB state and in the Recovery state (resp. Eqs. (**??**), (**??**) and Eqs. (**??**), (**??**)) yields the system of equations

$$\begin{aligned}
\pi_Z &= \kappa y_Z + \beta \pi_R, \\
y_Z &= y_R - (i_Z - y_R - r_Z) + \gamma b_Z, \\
\pi_R &= \kappa y_R + \beta p \pi_Z, \\
y_R &= p y_Z - (i_R - p\pi_Z - r_R) + \gamma b_R,
\end{aligned}$$

(Note that here and in the next results we are already making use of the zero inflation equilibrium outcomes under steady state.) Using the previous equations together with the monetary and fiscal optimal policies in the ZLB state (Eqs. (27), (24), (25), (26)),

$$i_Z = 0, \quad b_Z = \frac{\alpha_y^{Tr}\gamma}{\alpha_y^{Tr}\gamma^2 + \beta^{Tr}\alpha_\tau^{Tr}(1+r_R)^2}\left(i_Z - (y_Z^e + \pi_Z^e) - r_Z\right), \quad \tau_Z = 0, \quad \ell_Z = b_Z,$$

and in the Recovery state (Eqs. (22), (15), (17) (16)),

$$i_R = \gamma b_R + y_R^e + \left(\frac{\alpha_y^{CB} + \alpha_\pi^{CB}\kappa^2 + \alpha_\pi^{CB}\kappa\beta}{\alpha_y^{CB} + \alpha_\pi^{CB}\kappa^2}\right)\pi_R^e + r_R, \quad b_R = 0, \quad \tau_R = (1 + r_R)\, b_Z, \quad \ell_R = 0,$$

I can build a system of equations with solution (denoted with asterisks)

$$\begin{aligned}
\pi_Z^* &= \frac{\beta\kappa\alpha_\tau^{Tr}\left(\alpha_y^{CB} + \kappa^2\alpha_\pi^{CB}\right)(1+r_R)^2}{\phi} r_Z, & y_Z^* &= \frac{\beta\alpha_\tau^{Tr}\left(\left(1-p\beta^2\right)\alpha_y^{CB} + \kappa^2\alpha_\pi^{CB}\right)(1+r_R)^2}{\phi} r_Z, & i_Z^* &= 0, \\
b_Z^* &= -\frac{\gamma\alpha_y^{Tr}\left(\left(1-p\beta^2\right)\alpha_y^{CB} + \kappa^2\alpha_\pi^{CB}\right)}{\phi} r_Z & \ell_Z^* &= -\frac{\gamma\alpha_y^{Tr}\left(\left(1-p\beta^2\right)\alpha_y^{CB} + \kappa^2\alpha_\pi^{CB}\right)}{\phi} r_Z & \tau_Z^* &= 0
\end{aligned}$$

$$\begin{aligned}
\pi_R^* &= \frac{\beta p\kappa\alpha_y^{CB}\alpha_\tau^{Tr}\beta(1+r_R)^2}{\phi} r_Z, \qquad y_R^* = -\frac{\beta^2 p\kappa^2\alpha_\pi^{CB}\alpha_\tau^{Tr}(1+r_R)^2}{\phi} r_Z, \\
i_R^* &= \frac{\beta p\left(\left(1-p\beta^2+\kappa\right)\alpha_y^{CB} + (1+\kappa+\beta)\kappa^2\alpha_\pi^{CB}\right)\alpha_\tau^{Tr}(1+r_R)^2}{\phi} r_Z + r_R, \\
b_R^* &= 0 \qquad \ell_R^* = 0 \\
\tau_R^* &= -\frac{\gamma\alpha_y^{Tr}\left(\left(1-p\beta^2\right)\alpha_y^{CB} + \kappa^2\alpha_\pi^{CB}\right)(1+r_R)}{\phi} r_Z
\end{aligned}$$

and $\pi_S^* = 0$, $y_S^* = 0$, $i_S^* = r_S$, $b_S^* = 0$, $\ell_S^* = 0$, and $\tau_S^* = 0$, with $\phi \equiv \alpha_y^{CB}[(1 - p\beta^2)\gamma^2\alpha_y^{Tr} + (1 - p\beta\,(\beta + \kappa))\alpha_\tau^{Tr}\beta\,(1 + r_R)^2] + \alpha_\pi^{CB}\kappa^2(\gamma^2\alpha_y^{Tr} + (1 + \beta p)\,\alpha_\tau^{Tr}\beta\,(1 + r_R)^2)$. ■

Appendix B - Scenario with forward guidance

Proof of Propositions

The following proofs of propositions and notation closely follow Nakata (2018) and Nakata (2018) (with underlying logic based on Chari and Kehoe (1990)), except that here I adjust these to incorporate another policymaker (the treasury). In addition, I also follow Jung et al. (2005) to discuss uniqueness.

Proof of Proposition 6. Call the strategy profile $\left\{\sigma^{CB,*}, \sigma^{Tr,*}, \sigma^{PS,*}\right\}$ the solution under discretion. This strategy profile can be built as follows. Recall the discretionary monetary policy result, $\left\{i_j^*\right\}$, the fiscal policy results, $\left\{b_j^*, \tau_j^*, \ell_j^*\right\}$, and the private sector results, $\left\{\pi_j^*, y_j^*\right\}$, defined by Proposition (4) for each state $j = \{Z, R, S\}$. Be the strategy for the central bank $\sigma^{CB,*} = \left\{\sigma_t^{CB,*}\left(h_t^{CB}\right)\right\}_{t=0}^{\infty}$. I define each element of strategy $\sigma^{CB,*}$, $\sigma_t^{CB,*}\left(h_t^{CB}\right)$, using the results for monetary policy from Proposition (4) $\forall t \geq 0$ and $\forall h_t$ as

$$\sigma_t^{CB,*}\left(h_t^{CB}\right) = i_k^* \text{ if } r_t = r_k, \text{ with } k = \{Z, R, S\}. \tag{103}$$

Similarly, for the treasury define $\sigma^{Tr,*} = \left\{\sigma_t^{Tr,*}\left(h_t^{Tr}\right)\right\}_{t=0}^{\infty}$, with each element of following fiscal policy results from Proposition (4) $\forall t \geq 0$ and $\forall h_t$ as

$$\sigma_t^{Tr,*}\left(h_t^{Tr}\right) = \left(b_k^*, \tau_k^*, \ell_k^*\right) \text{ if } r_t = r_k, \text{ with } k = \{Z, R, S\}. \tag{104}$$

Finally, be the strategy for the Private Sector $\sigma^{PS,*} = \left\{\sigma_t^{PS*}\left(h_t^{PS}\right)\right\}_{t=0}^{\infty}$. Then, for each element of strategy $\sigma^{PS,*}$ it follows that, $\forall t \geq 0$ and $\forall h_t$ as

$$\sigma_t^{PS,*}\left(h_t^{PS}\right) = \left(\pi_k^*, y_k^*\right) \text{ if } r_t = r_k, \text{ with } k = \{Z, R, S\}. \tag{105}$$

I seek to argue that the equilibrium outcome under discretion is a SE; i.e., that $\left\{\sigma^{CB,*}, \sigma^{Tr,*}, \sigma^{PS,*}\right\}$ satisfies items 1-3 from the SE definition. Formally, I want to show that, for any current history, h_t^k with $k = \{CB, Tr, PS\}$, each continuation strategy $\sigma_t^{k,*}$ for $k = \{CB, Tr, PS\}$ is a best response to strategy $\sigma^{-k,*}$. Let us show first that the central bank's continuation strategy that can be formed with strategy $\sigma^{CB,*}$ with element specification in Eq. (103) satisfies the SE definition (item 2). To show this, it is sufficient to apply the One-Shot Deviation Principle and rule out any profitable deviations. For every t and history h_t, I want to show that there is no welfare improving deviation compared to the discretionary outcome; this means showing that

$$l_t^{CB}\left(y_t\left(i_t^*\right), \pi_t\left(i_t^*\right)\right) + \beta E_t L_{t+1}^{CB} \geq l_t^{CB}\left(y_t\left(i_t\right), \pi_t\left(i_t\right)\right) + \beta E_t L_{t+1}^{CB}$$

for every i_t such that $i_t \neq i_t^*$. Due to the nature of the discretionary problem, we know that the continuation values $\beta E_t L_{t+1}^{CB}$ at both sides of the inequality do not depend on the current choice of i_t. Since the discretionary outcome of the left-hand side comes defined by the nominal interest rate that maximizes the stage payoff $l_t^{CB}(\cdot)$, i.e.,

$$\max_{i_t} l_t^{CB}\left(y_t\left(i_t\right), \pi_t\left(i_t\right)\right) + \beta E_t L_{t+1}^{CB} \equiv l_t^{CB}\left(y_t\left(i_t^*\right), \pi_t\left(i_t^*\right)\right) + \beta E_t L_{t+1}^{CB} \geq l_t^{CB}\left(y_t\left(i_t\right), \pi_t\left(i_t\right)\right) + \beta E_t L_{t+1}^{CB}$$

then the inequality shown above holds. Therefore, $\sigma_t^{CB,*}\left(h_t^{CB}\right) = i_k^*$ if $r_t = r_k$ with $k = \{Z, R, S\}$, and since this holds for every element of the continuation strategy $\sigma_t^{CB,*}$, then the discretionary specification in Eq. (**??**) satisfies item 2 of the SE definition.

For the treasury's case, we want to show that its continuation strategy formed with strategy $\sigma^{Tr,*}$ with element specification in Eq. (104) satisfies the SE definition (item 3). The proof follows a similar logic as that of the CB, with the only difference being that the treasury solves problem (13). Note that, since the central bank leads the treasury, then i_t^* is such that it internalizes the treasury's response. As such, it rules out any incentives for the treasury to deviate – if this were not the case, this would mean that the central bank did not foresee the treasury's response (which is ruled out because of the timing of the moves), or that the central bank is not maximizing its payoff from the very beginning. Therefore, after i_t^* is chosen, the treasury must be better off when it plays on the equilibrium path; i.e., $\sigma_t^{Tr,*}\left(h_t^{Tr}\right) = (b_R^*, \tau_R^*, \ell_R^*)$ if $r_t = r_k$ with $k = \{Z, R, S\}$. This holds for every element of the continuation strategy $\sigma_t^{Tr,*}$, and it implies that the discretionary specification in Eq. (104) satisfies the SE definition (item 3).

For the Private Sector, we need to show that its continuation strategy formed with strategy $\sigma^{PS,*}$ with element specification in Eq. (**??**) satisfies the SE definition (item 1). We therefore want to show that $\left(\sigma_t^{PS,\pi,}(r_k), \sigma_t^{PS,y}(r_k)\right)$ yields (π_k^*, y_k^*) for every k-state. Build the system of equations using the DIS and NKPC from Expression (**??**) at Z and R,

$$\sigma_t^{PS,\pi}(r_Z) = \kappa\sigma_t^{PS,y}(r_Z) + \beta E_t\left[\sigma_{t+1}^{PS,\pi}(r_Z)\right],$$

$$\sigma_t^{PS,y}(r_Z) = E_t\left[\sigma_{t+1}^{PS,y}(r_Z)\right] - \left(i_Z^* - E_t\left[\sigma_{t+1}^{PS,\pi}(r_Z)\right] - r_Z\right) + \gamma b_Z^*,$$

$$\sigma_t^{PS,\pi}(r_R) = \kappa\sigma_t^{PS,y}(r_R) + \beta E_t\left[\sigma_{t+1}^{PS,\pi}(r_R)\right],$$

$$\sigma_t^{PS,y}(r_R) = E_t\left[\sigma_{t+1}^{PS,y}(r_R)\right] - \left(i_R^* - E_t\left[\sigma_{t+1}^{PS,\pi}(r_R)\right] - r_R\right) + \gamma b_R^*,$$

and invoke rational expectations,

$$E_t\left[\sigma_{t+1}^{PS,\pi}(r_Z)\right] = \sigma_{t+1}^{PS,\pi}(r_R), \qquad E_t\left[\sigma_{t+1}^{PS,y}(r_Z)\right] = \sigma_{t+1}^{PS,y}(r_R),$$

$$E_t\left[\sigma_{t+1}^{PS,\pi}(r_R)\right] = p\sigma_t^{PS,\pi}(r_Z), \qquad E_t\left[\sigma_{t+1}^{PS,y}(r_R)\right] = p\sigma_t^{PS,y}(r_Z).$$

This yields

$$\sigma_t^{PS,\pi}(r_Z) = \kappa\sigma_t^{PS,y}(r_Z) + \beta\sigma_{t+1}^{PS,\pi}(r_R)$$

$$\sigma_t^{PS,y}(r_Z) = \sigma_{t+1}^{PS,y}(r_R) - \left(i_Z^* - \sigma_{t+1}^{PS,\pi}(r_R) - r_Z\right) + \gamma b_Z^*,$$

$$\sigma_t^{PS,\pi}(r_R) = \kappa\sigma_t^{PS,y}(r_R) + \beta p\sigma_t^{PS,\pi}(r_Z),$$

$$\sigma_t^{PS,y}(r_R) = p\sigma_t^{PS,y}(r_Z) - \left(i_R^* - p\sigma_t^{PS,\pi}(r_Z) - r_R\right) + \gamma b_R^*.$$

Solving further the system of equations and plugging in the expressions for $(i_Z^*, b_Z^*, \tau_Z^*, \ell_Z^*)$ and $(i_R^*, b_R^*, \tau_R^*, \ell_R^*)$ from Proposition (4) yields

$$\sigma_t^{PS,\pi}(r_Z) = \theta_Z^\pi r_Z \qquad \sigma_t^{PS,y}(r_Z) = \theta_Z^y r_Z \qquad \sigma_t^{PS,\pi}(r_R) = \theta_R^\pi r_Z \qquad \sigma_t^{PS,y}(r_R) = \theta_R^y r_Z$$

which are the exact expressions for (π_Z^*, y_Z^*) and (π_R^*, y_R^*) from the same Proposition (and, therefore, from Eq. (105)).

The solutions for the central bank and the treasury are unique. In addition, it is also easily verified that the solution for the Private sector is unique. If we impose $\left\{\sigma^{CB,*}, \sigma^{Tr,*}\right\}$, we want to show for the PS that the solution yields the previous expressions $\sigma_t^{PS,\pi}\left(h_t^{PS}\right) = \theta_k^{\pi} r_k$ and $\sigma_t^{PS,y}\left(h_t^{PS}\right) = \theta_k^{y} r_k$ (for $k = \{Z, R\}$) in strategies $\sigma_s^{PS,\pi}$ and $\sigma_s^{PS,y}$ for any $s > t$. Iterating forward equations DIS (from Expression (**??**)) and NKPC from (Expression (**??**)) for any $s > t$ and h_s^{PS} (and abstracting from arguments below), yields

$$\sigma_s^{PS,y} = -E_s\left\{\sum_{k=0}^{\infty} i_{s+j} - \sigma_{s+1+j}^{PS,\pi} - r_{s+j} + \gamma b_{s+j}\right\}$$

yields

$$\sigma_s^{PS,\pi} = \kappa E_s\left\{\sum_{k=0}^{\infty} \beta^j \sigma_{s+j}^{PS,y}\right\}$$

If the economy starts at Z,

$$\sigma_Z^{PS,y} = -E_s\left\{\sum_{k=0}^{\infty} i_{s+j} - \sigma_{s+1+j}^{PS,\pi} - r_{s+j} + \gamma b_{s+j}\right\} = \frac{1}{1-p}\left[\sigma_R^{PS,\pi} + r_Z + \gamma b_Z - i_R + p\sigma_Z^{PS,\pi} + r_R\right]$$

$$\sigma_Z^{PS,\pi} = \kappa E_s\left\{\sum_{k=0}^{\infty} \beta^j \sigma_{s+j}^{PS,y}\right\} = \frac{\kappa}{1-\beta^2 p}\left(\sigma_Z^{PS,y} + \beta\sigma_R^{PS,y}\right)$$

and if the economy starts at R,

$$\sigma_R^{PS,y} = -E_s\left\{\sum_{k=0}^{\infty} i_{s+j} - \sigma_{s+1+j}^{PS,\pi} - r_{s+j} + \gamma b_{s+j}\right\} = \frac{1}{1-p}\left[p\left(\sigma_R^{PS,\pi} + r_Z + \gamma b_Z\right) - \left(i_R - p\sigma_Z^{PS,\pi} - r_R\right)\right]$$

$$\sigma_R^{PS,\pi} = \kappa E_s\left\{\sum_{k=0}^{\infty} \beta^j \sigma_{s+j}^{PS,y}\right\} = \frac{\kappa}{1-\beta^2 p}\left(\sigma_R^{PS,y} + \beta p\sigma_Z^{PS,y}\right)$$

(where for simplicity I imposed that $b_R = 0$ and $i_Z = 0$). (In the two expressions from above we simplified the infinite summation by assuming that the PS chooses the same strategy.) We obtained the same system of equations outlined for the period-t strategies from before. Hence, plugging in the expressions for $(i_Z^*, b_Z^*, \tau_Z^*, \ell_Z^*)$ and $(i_R^*, b_R^*, \tau_R^*, \ell_R^*)$ from Proposition (4) yields the same exact expressions for (π_Z^*, y_Z^*) and (π_R^*, y_R^*) from Proposition (4). Therefore, $\sigma_t^{PS,y}\left(h_t^{PS}\right) = \sigma_s^{PS,y}\left(h_s^{PS}\right) \forall s > t$, and the solution is unique.

We now want to prove the statement that $\sigma^* = \left\{\sigma^{CB,*}, \sigma^{Tr,*}, \sigma^{PS,*}\right\}$ is the worst SE. We do this by contradiction – see, for instance, Kurozumi (2008) Assuming instead that there exists a $\hat{\sigma} = \left\{\hat{\sigma}^{CB}, \hat{\sigma}^{Tr}, \hat{\sigma}^{PS}\right\}$ worse than σ^*. If that is the case, then CB losses at any t, history h_t^{CB}, are

$$V^{CB,*}\left(h_t, \sigma^{CB,*}, \sigma^{Tr,*}, \sigma^{PS,*}\right) > \hat{V}^{CB}\left(h_t, \hat{\sigma}^{CB}, \hat{\sigma}^{Tr}, \hat{\sigma}^{PS}\right)$$

$$
\begin{aligned}
&-\tfrac{1}{2}E_t\left\{\sum_{k=t}^{\infty}\beta^{k-t}\left[\alpha_y^{CB}\left[\sigma_k^{PS,y,*}\left(h_k,\left(\sigma_k^{CB,*}\left(h_k^{CB}\right),\sigma_k^{Tr,*}\left(h_k,\sigma_k^{CB,*}\left(h_k^{CB}\right)\right)\right)\right)\right]^2+\right.\right.\\
&\left.\left.\alpha_\pi^{CB}\left[\sigma_k^{PS,\pi,*}\left(h_k,\left(\sigma_k^{CB,*}\left(h_k^{CB}\right),\sigma_k^{Tr,*}\left(h_k,\sigma_k^{CB,*}\left(h_k^{CB}\right)\right)\right)\right)\right]^2\right]\right\}>\\
&-\tfrac{1}{2}E_t\left\{\sum_{k=t}^{\infty}\beta^{k-t}\left[\alpha_y^{CB}\left[\hat{\sigma}_k^{PS,y}\left(h_k,\left(\hat{\sigma}_k^{CB}\left(h_k^{CB}\right),\hat{\sigma}_k^{Tr}\left(h_k,\hat{\sigma}_k^{CB}\left(h_k^{CB}\right)\right)\right)\right)\right]^2+\right.\right.\\
&\left.\left.\alpha_\pi^{CB}\left[\hat{\sigma}_k^{PS,\pi}\left(h_k,\left(\hat{\sigma}_k^{CB}\left(h_k^{CB}\right),\hat{\sigma}_k^{Tr}\left(h_k,\hat{\sigma}_k^{CB}\left(h_k^{CB}\right)\right)\right)\right)\right]^2\right]\right\}
\end{aligned}
\tag{106}
$$

(where DIS, NKPC and ZLB and treasury's policy function are skipped but these are satisfied in every period).

Recall that, if $\hat{\sigma}=\left\{\hat{\sigma}^{CB},\hat{\sigma}^{Tr},\hat{\sigma}^{PS}\right\}$ is the worst SE, it means that there is no profitable deviation from it for every history h_t – and, in particular, it means that σ^* is not a deviation too. Since strategy $\hat{\sigma}$ is a very generic element, we use σ^* to build how any candidate description of $\hat{\sigma}$ (but different from σ^*) may look. There are two groups formed by different candidate descriptions for $\hat{\sigma}$, where $\hat{\sigma}^{CB}$ is the strategy that will allow deviations (since σ^{Tr} and σ^{PS} are followers and their deviations should be internalized by $\hat{\sigma}^{CB}$ if $\hat{\sigma}^{CB}$ is an equilibrium). Also note that all strategies below for $\hat{\sigma}$ will be defined in terms of $\hat{\sigma}^{CB}$ only, leaving $\hat{\sigma}^{Tr}$ and $\hat{\sigma}^{PS}$ unspecified. The first group has one strategy description for $\hat{\sigma}$ given by

$$
\hat{\sigma}_t^{CB}\left(h_t^{CB}\right)=\sigma_t^{CB,*}\left(h_t^{CB}\right) \tag{107}
$$

for every t, given history h_t and every future history induced by $\hat{\sigma}$. The second group (with infinite strategy descriptions except from the strategy described in the first group) is defined by some $\tilde{t},h_{\tilde{t}}$ and future history induced by $\hat{\sigma}$ such that

$$
\hat{\sigma}_t^{CB}\left(h_t^{CB}\right)\begin{cases}=\sigma_t^{CB,*}\left(h_t^{CB}\right) & ,\ 0\leq t\leq\tilde{t}-1\\ \neq\sigma_t^{CB,*}\left(h_t^{CB}\right) & ,\ t=\tilde{t}\\ \text{free} & ,\ t>\tilde{t}\end{cases} \tag{108}
$$

If we can show that us inequality from Eq. (106) does not hold for any $\hat{\sigma}$ prescribed by the whole set of possible candidates portrayed in group 1 (Eq. (107)) and 2 (Eq. (108)), then we will have proven that σ^* is the worst SE by contradiction.

I shall start with the second group first (the first one will become trivial after that). Specifically, I want to show that there is a profitable deviation from strategy profiles $\hat{\sigma}$ that follow Eq. (108). I focus on a deviation of the Central bank strategy to $\sigma^{CB,dev}$ – yet to be determined. Since $\hat{\sigma}$ is a Sustainable Equilibrium by assumption, we know that it must be such that $\forall t,h_t$,

$$
\hat{V}^{CB}\left(h_t,\hat{\sigma}^{CB},\hat{\sigma}^{Tr},\hat{\sigma}^{PS}\right)\geq\hat{V}^{CB}\left(h_t,\sigma^{CB,dev},\hat{\sigma}^{Tr},\hat{\sigma}^{PS}\right), \tag{109}
$$

with $\hat{V}^{CB}\left(h_t,\sigma^{CB,dev},\hat{\sigma}^{Tr},\hat{\sigma}^{PS}\right)$ given by

$$
\begin{aligned}
&-\tfrac{1}{2}E_t\left\{\sum_{k=t}^{\infty}\beta^{k-t}\left[\alpha_y^{CB}\left[\hat{\sigma}_k^{PS,y}\left(h_k,\left(\sigma_k^{CB,dev}\left(h_k^{CB}\right),\hat{\sigma}_k^{Tr}\left(h_k,\hat{\sigma}_k^{CB}\left(h_k^{CB}\right)\right)\right)\right)\right]^2+\right.\right.\\
&\left.\left.\alpha_\pi^{CB}\left[\hat{\sigma}_k^{PS,\pi}\left(h_k,\left(\sigma_k^{CB,dev}\left(h_k^{CB}\right),\hat{\sigma}_k^{Tr}\left(h_k,\hat{\sigma}_k^{CB}\left(h_k^{CB}\right)\right)\right)\right)\right]^2\right]\right\}
\end{aligned}
$$

(where DIS, NKPC and ZLB and treasury's policy function are satisfied in every period). Consider now a deviation from Eq. (108) to $\sigma_t^{CB,dev}\left(h_t^{CB}\right)=\sigma_t^{CB,*}\left(h_t^{CB}\right)$. But under this specific deviation, we know that

$$\hat{\sigma}_k^{Tr}\left(h_k,\hat{\sigma}_k^{CB}(\cdot)\right)=\hat{\sigma}_k^{Tr}\left(h_k,\sigma_k^{CB,dev}(\cdot)\right)=\hat{\sigma}_k^{Tr}\left(h_k,\sigma_k^{CB,*}(\cdot)\right)=\sigma_k^{Tr,*}\left(h_k,\sigma_k^{CB,*}(\cdot)\right).$$

Likewise, PS will observe both actions, and its unique best response is $\sigma^{PS,*}$. I.e., formally we would obtain that $\hat{\sigma}_k^{PS}\left(h_k,\hat{\sigma}_k^{CB}(\cdot),\hat{\sigma}_k^{Tr}\left(h_k,\hat{\sigma}_k^{CB}(\cdot)\right)\right)$ is given by $\sigma_k^{PS,*}\left(h_k,\sigma_k^{CB,*}(\cdot),\sigma_k^{Tr,*}\left(h_k,\sigma_k^{CB,*}(\cdot)\right)\right)$. Plugging in the RHS of Eq. (109) the result for the PS inside the history observed by the treasury, and doing the same with this new object inside the history of the CB the exact same present discounted value for the central bank under discretion, $V^{CB,*}\left(h_t,\sigma^{CB,*},\sigma^{Tr,*},\sigma^{PS,*}\right)$. Finally, taking this into Eq. (109) yields

$$\hat{V}^{CB}\left(h_t,\hat{\sigma}^{CB},\hat{\sigma}^{Tr},\hat{\sigma}^{PS}\right)\geq\hat{V}^{CB}\left(h_t,\sigma^{CB,dev},\hat{\sigma}^{Tr},\hat{\sigma}^{PS}\right)=V^{CB,*}\left(h_t,\sigma^{CB,*},\sigma^{Tr,*},\sigma^{PS,*}\right).$$

But linking this chain of inequalities in our initial inequality (Eq. (106)), renders

$$V^{CB,*}\left(h_t,\sigma^{CB,*},\sigma^{Tr,*},\sigma^{PS,*}\right)>V^{CB,*}\left(h_t,\sigma^{CB,*},\sigma^{Tr,*},\sigma^{PS,*}\right)$$

which is a contradiction. Hence, any candidate strategy $\widehat{\sigma}$ described as per group 2 (Eq. (108)) cannot do worse than σ^*. Following the same procedure with group 1 (Eq. (107)), it is easy to see that we obtain the same inequality as the one above – this is immediate because the candidate strategy $\widehat{\sigma}$ is exactly equal to σ^*. Therefore, there is no alternative strategy profile $\hat{\sigma}=\left\{\hat{\sigma}^{CB},\hat{\sigma}^{Tr},\hat{\sigma}^{PS}\right\}$ that does worse than the discretionary equilibrium strategy profile, $\sigma^*=\left\{\sigma^{CB,*},\sigma^{Tr,*},\sigma^{PS,*}\right\}$ and, hence, strategy profile σ^* is the worst SE. ■

Proof of Proposition 7. ($\Rightarrow$) We want to show that if $\sigma^S=\left\{\sigma^{CB,S},\sigma^{Tr,S},\sigma^{PS,S}\right\}$ is a SE with outcome $\left(i_t^S,b_t^S,y_t^S,\pi_t^S\right)$, then it satisfies items 1,2,3 of the Proposition 7.

By assumption, if $\sigma^S=\left\{\sigma^{CB,S},\sigma^{Tr,S},\sigma^{PS,S}\right\}$ is a SE, then as per the definition of SE we know that the continuation strategy of the private sector, $\sigma_t^{PS,S}$, satisfies rational expectations and the NKPC and the DIS for every current history h_t^{PS}. Hence, the outcome $\left(i_t^S,b_t^S,y_t^S,\pi_t^S\right)$ satisfies a PSCE (and, therefore, item 1 of the proposition is satisfied). Following a similar logic, we can prove item 2 too. To prove item 3, we follow Chari et al. (1998) and Kurozumi (2008), and assume, on the contrary, that there exists a $\tilde{t}$ such that the Sustainability Constraint does not hold; i.e.,

$$V_{\tilde{t}}^{CB,S}\left(h_{\tilde{t}},\sigma^{CB,S},\sigma^{Tr,S},\sigma^{PS,S}\right)<V_{\tilde{t}}^{CB,*}\left(h_{\tilde{t}},\sigma^{CB,*},\sigma^{Tr,*},\sigma^{PS,*}\right)\tag{110}$$

with $V_{\tilde{t}}^{CB,*}\left(h_{\tilde{t}},\sigma^{CB,*},\sigma^{Tr,*},\sigma^{PS,*}\right)$ given by

$$-\frac{1}{2}E_t\left\{\sum_{k=t}^{\infty}\beta^{k-t}\left[\alpha_y^{CB}\left[\sigma_k^{PS,*}\left(h_k,\left(\sigma_k^{CB,*}\left(h_k^{CB}\right),\sigma_k^{Tr,*}\left(h_k,\sigma_k^{CB,*}\left(h_k^{CB}\right)\right)\right)\right)\right]^2+\right.\right.$$
$$\left.\left.\alpha_\pi^{CB}\left[\sigma_k^{PS,*}\left(h_k,\left(\sigma_k^{CB,*}\left(h_k^{CB}\right),\sigma_k^{Tr,*}\left(h_k,\sigma_k^{CB,*}\left(h_k^{CB}\right)\right)\right)\right)\right]^2\right]\right\}$$

and $V_{\tilde{t}}^{CB,S}\left(h_{\tilde{t}},\sigma^{CB,S},\sigma^{Tr,S},\sigma^{PS,S}\right)$ defined by

$$-\frac{1}{2}E_t\left\{\sum_{k=t}^{\infty}\beta^{k-t}\left[\alpha_y^{CB}\left[\sigma_k^{PS,y,S}\left(h_k,\left(\sigma_k^{CB,S}\left(h_k^{CB}\right),\sigma_k^{Tr,S}\left(h_k,\sigma_k^{CB,S}\left(h_k^{CB}\right)\right)\right)\right)\right]^2+\right.\right.$$
$$\left.\left.\alpha_\pi^{CB}\left[\sigma_k^{PS,\pi,S}\left(h_k,\left(\sigma_k^{CB,S}\left(h_k^{CB}\right),\sigma_k^{Tr,S}\left(h_k,\sigma_k^{CB,*}\left(h_k^{CB}\right)\right)\right)\right)\right]^2\right]\right\}$$

By definition of SE, since $\sigma^S=\left\{\sigma^{CB,S},\sigma^{Tr,S},\sigma^{PS,S}\right\}$ is a SE, then it has to satisfy that $\sigma^{CB,S}$ renders a payoff weakly better than any deviation to $\sigma^{CB,dev}$ $\forall t,h_t$ – provided that the other players are following the SE prescription, $\left\{\sigma^{Tr,S},\sigma^{PS,S}\right\}$. Formally, for $t\geq\tilde{t}$, this means that for any deviation to $\sigma^{CB,dev}$,

$$V_{\tilde{t}}^{CB,S}\left(h_{\tilde{t}},\sigma^{CB,S},\sigma^{Tr,S},\sigma^{PS,S}\right)\geq V_{\tilde{t}}^{CB,S}\left(h_{\tilde{t}},\sigma^{CB,dev},\sigma^{Tr,S},\sigma^{PS,S}\right)\tag{111}$$

Consider the particular case of a deviation $\sigma_{\tilde{t}}^{CB,dev}=\sigma_{\tilde{t}}^{CB,*}$ – for which the value of V is known. From the proof of Proposition 1, we know that the best response from the treasury to that deviation instructs the treasury to do $\sigma_t^{Tr,S}\left(h_{t,}\sigma_t^{CB,dev}(\cdot)\right)=\sigma_t^{Tr,*}\left(h_{t,}\sigma_t^{CB,*}(\cdot)\right)$, and for the Private sector, $\sigma_t^{PS,S}\left(h_t^{CB},\sigma_t^{CB,dev}(\cdot),\sigma_t^{Tr,S}\left(h_{t,}\sigma_t^{CB,dev}(\cdot)\right)\right)=\sigma_t^{PS,*}\left(h_{t,}\sigma_t^{CB,*}(\cdot),\sigma_t^{Tr,*}\left(h_{k,}\sigma_t^{CB,*}(\cdot)\right)\right)$. Plugging these in the RHS of Eq. (111) yields $V_{\tilde{t}}^{CB,*}\left(h_{\tilde{t}},\sigma^{CB,*},\sigma^{Tr,*},\sigma^{PS,*}\right)$. Taking this result to Eq. (111) it means that

$$V_{\tilde{t}}^{CB,S}\left(h_{\tilde{t}},\sigma^{CB,S},\sigma^{Tr,S},\sigma^{PS,S}\right)\geq V_{\tilde{t}}^{CB,S}\left(h_{\tilde{t}},\sigma^{CB,dev},\sigma^{Tr,S},\sigma^{PS,S}\right)=V_{\tilde{t}}^{CB,*}\left(h_{\tilde{t}},\sigma^{CB,*},\sigma^{Tr,*},\sigma^{PS,*}\right)$$

But plugging back this chain of inequalities in our initial statement again (Eq. (110)), renders

$$V_{\tilde{t}}^{CB,*}\left(h_{\tilde{t}},\sigma^{CB,*},\sigma^{Tr,*},\sigma^{PS,*}\right)<V_{\tilde{t}}^{CB,*}\left(h_{\tilde{t}},\sigma^{CB,*},\sigma^{Tr,*},\sigma^{PS,*}\right)$$

(a contradiction). Hence, it can only be that Eq. (110) holds as

$$V_{\tilde{t}}^{CB,S}\left(h_{\tilde{t}},\sigma^{CB,S},\sigma^{Tr,S},\sigma^{PS,S}\right)\geq V_{\tilde{t}}^{CB,*}\left(h_{\tilde{t}},\sigma^{CB,*},\sigma^{Tr,*},\sigma^{PS,*}\right)$$

which proves item 3 of the proposition (the Sustainability constraint).

($\Leftarrow$) If $\left(i_t^S,b_t^S,\tau_t^S,\ell_t^S,y_t^S,\pi_t^S\right)$ satisfy items 1,2,3 from Proposition 7, then $\left(i_t^S,b_t^S,\tau_t^S,\ell_t^S,y_t^S,\pi_t^S\right)$ is an outcome of a SE. We prove this by construction. Specifically, following Kurozumi (2008), we recall the grim trigger strategy $\sigma^S=\left\{\sigma^{CB,S},\sigma^{Tr,S},\sigma^{PS,S}\right\}$ that supports outcomes $\left(i_t^S,b_t^S,\tau_t^S,\ell_t^S,y_t^S,\pi_t^S\right)$ like the one shown in text – see also Basso (2009) for a similar proof.

Let us analyze first what happens if the central bank did not deviate. Since item 1 of Prop 2 is satisfied by $\left(i_t^S,b_t^S,\tau_t^S,\ell_t^S,y_t^S,\pi_t^S\right)$ (by assumption), then it is a PSCE, and this satisfies the Private Sector part of the SE definition. The same applies for item 2 of Prop 2. Finally, since the central bank did not deviate, we know that $\left(i_t^S,b_t^S,\tau_t^S,\ell_t^S,y_t^S,\pi_t^S\right)$ is realized and it satisfies item 3 of Prop 2 (the SC) (by assumption). So, the Central bank must be maximizing its payoff, which matches the central bank part of the SE definition ($\left(i_t^S,b_t^S,\tau_t^S,\ell_t^S,y_t^S,\pi_t^S\right)$ is satisfying the CB's maximization problem given the constraints).

Now let us analyze what happens if the central bank deviated at some point in the history of past or current events. If there existed a deviation, then the trigger strategy instructs (y_t^*,π_t^*) to be chosen by the PS, and the CB and the treasury's best responses are i_t^* and $(b_t^*,\tau_t^*,\ell_t^*)$, which is a PSCE (we already showed that before, see Proposition 1). For the treasury, upon central bank's deviation, b_t^* will be chosen, which also satisfies treasury's policy (the PS's and the treasury's best responses are (y_t^*, π_t^*) and b_t^*). Finally, since the CB deviated, then it has to set i_t^*, and since $(i_t^*, b_t^*, \tau_t^*, \ell_t^*, y_t^*, \pi_t^*)$ is realized, the CB attains $V^{CB,*}$. Since $V^{CB,*}$ is a value such that it emerges when all central bank, treasury and PS optimize (under discretion), it is an equilibrium. Furthermore, since by Proposition 1 we showed that the discretionary equilibrium is a SE, then we also prove that this grim trigger strategy yields outcomes belong to a SE. So, we proved by construction that this strategy sustains an outcome $\left(i_t^S, b_t^S, \tau_t^S, \ell_t^S, y_t^S, \pi_t^S\right)$ that is a SE. ∎

Lemma 17 *Under discretion, and given the central bank's time-invariant losses* $\{l_Z^{CB}\left(y_Z^*, \pi_Z^*\right), l_R^{CB}\left(y_R^*, \pi_R^*\right), l_S^{CB}\left(y\right.$ *the present discounted value of losses in periods where the state of the world is* Z, $V_Z^{CB,*}$, *is determined by*

$$V_Z^{CB,*} = \frac{1}{1-\beta^2 p}\left(l_Z^{CB}\left(y_Z^*, \pi_Z^*\right) + \beta l_R^{CB}\left(y_R^*, \pi_R^*\right)\right) + \frac{1-p}{1-\beta^2 p}\frac{\beta^2}{1-\beta} l_S^{CB}\left(y_S^*, \pi_S^*\right).$$

Proof of 17. Consider the first periods (and group by state) to get

$$\begin{aligned}
V_Z^{CB,*} = \; & l_Z^{CB}\left(y_Z^*, \pi_Z^*\right) + \beta^2 p l_Z^{CB}\left(y_Z^*, \pi_Z^*\right) + \beta^4 p^2 l_Z^{CB}\left(y_Z^*, \pi_Z^*\right) + \beta^6 p^3 l_Z^{CB}\left(y_Z^*, \pi_Z^*\right) + \\
& \beta l_R^{CB}\left(y_R^*, \pi_R^*\right) + \beta^3 p l_R^{CB}\left(y_R^*, \pi_R^*\right) + \beta^5 p^2 l_R^{CB}\left(y_R^*, \pi_R^*\right) + \beta^7 p^3 l_R^{CB}\left(y_R^*, \pi_R^*\right) + \\
& \beta^8 p^4 [...] + \beta^8 p^3 (1-p) [...] + \beta^6 p^2 (1-p) \beta^2 [...] + \beta^4 p (1-p) \beta^4 [...] + \beta^2 (1-p) \beta^6 [...] + \\
& \beta^6 p^2 (1-p) l_S^{CB}\left(y_S^*, \pi_S^*\right) + \beta^6 p^2 (1-p) \beta l_S^{CB}\left(y_S^*, \pi_S^*\right) + \beta^4 p (1-p) l_S^{CB}\left(y_S^*, \pi_S^*\right) + \\
& \beta^4 p (1-p) \beta l_S^{CB}\left(y_S^*, \pi_S^*\right) + \beta^4 p (1-p) \beta^2 l_S^{CB}\left(y_S^*, \pi_S^*\right) + \beta^4 p (1-p) \beta^3 l_S^{CB}\left(y_S^*, \pi_S^*\right) + \\
& \beta^2 (1-p) l_S^{CB}\left(y_S^*, \pi_S^*\right) + \beta^2 (1-p) \beta l_S^{CB}\left(y_S^*, \pi_S^*\right) + \beta^2 (1-p) \beta^2 l_S^{CB}\left(y_S^*, \pi_S^*\right) \\
& + \beta^2 (1-p) \beta^3 l_S^{CB}\left(y_S^*, \pi_S^*\right) + \beta^2 (1-p) \beta^4 l_S^{CB}\left(y_S^*, \pi_S^*\right) + \beta^2 (1-p) \beta^5 l_S^{CB}\left(y_S^*, \pi_S^*\right)
\end{aligned}$$

This recursion yields

$$V_Z^{CB,*} = \frac{1}{1-\beta^2 p}\left(l_Z^{CB}\left(y_Z^*, \pi_Z^*\right) + \beta l_R^{CB}\left(y_R^*, \pi_R^*\right)\right) + \frac{1-p}{1-\beta^2 p}\frac{\beta^2}{1-\beta} l_S^{CB}\left(y_S^*, \pi_S^*\right).$$

∎

Corollary 18 *The present discounted value of losses for the central bank under discretion in state* R *is*

$$V_R^{CB,*} \equiv \frac{l_R^{CB}\left(y_R^*, \pi_R^*\right) + \beta p l_Z^{CB}\left(y_Z^*, \pi_Z^*\right)}{1-\beta^2 p}$$

Proof of 18. To start at R, we have to subtract $l_Z^{CB}\left(y_Z^*, \pi_Z^*\right)$ and divide by β expression $V_Z^{CB,*}$ from Lemma 17 to stand from CB's perspective at R. Hence, (and abstracting from arguments)

$$V_R^{CB,*} = \frac{V_Z^{CB,*} - l_Z^{CB}}{\beta} = \frac{\frac{1}{1-\beta^2 p}\left(l_Z^{CB} + \beta l_R^{CB}\right) - l_Z^{CB}}{\beta} = \frac{\beta p l_Z^{CB}}{1-\beta^2 p} + \frac{l_R^{CB}}{1-\beta^2 p} = \frac{l_R^{CB} + \beta p l_Z^{CB}}{1-\beta^2 p}$$

∎

Corollary 19 *The discounted value of the CB's losses in the Recovery state can be reexpressed as*

$$V_R^{CB,*} \equiv l_R^{CB}\left(y_R^*, \pi_R^*\right) + \beta p \frac{l_Z^{CB}\left(y_Z^*, \pi_Z^*\right) + \beta l_R^{CB}\left(y_R^*, \pi_R^*\right)}{1 - \beta^2 p}$$

Proof of 19 . Using Corollary 18, then (and abstracting from arguments)

$$V_R^{CB,*} \equiv \frac{l_R^{CB} + \beta p l_Z^{CB}}{1 - \beta^2 p} = \frac{l_R^{CB}}{1 - \beta^2 p} + \frac{\beta p l_Z^{CB}}{1 - \beta^2 p} = l_R^{CB} + \frac{\beta p \left(l_Z^{CB} + \beta l_R^{CB}\right)}{1 - \beta^2 p}$$

■

Proof: Sustainability constraint re-expressed.. Note that (and skipping arguments momentarily to simplify the exposition)

$$L_N^{cb}\left(\widetilde{y}_N, \widetilde{\pi}_N\right) + \beta p \frac{L_Z^{cb}\left(\widetilde{y}_Z, \widetilde{\pi}_Z\right) + \beta L_N^{cb}\left(\widetilde{y}_N, \widetilde{\pi}_N\right)}{1 - \beta^2 p} \geq L_N^{cb}\left(y_N^*, \pi_N^*\right) + \beta p \frac{L_Z^{cb}\left(y_Z^*, \pi_Z^*\right) + \beta L_N^{cb}\left(y_N^*, \pi_N^*\right)}{1 - \beta^2 p}$$

$$\beta p \left[L_Z^{cb}\left(\widetilde{y}_Z, \widetilde{\pi}_Z\right) - L_Z^{cb}\left(y_Z^*, \pi_Z^*\right)\right] \geq L_N^{cb}\left(y_N^*, \pi_N^*\right) - L_N^{cb}\left(\widetilde{y}_N, \widetilde{\pi}_N\right)$$

■

Corollary 20 *If SC is satisfied with strict inequality, there exist multiple equilibria.*

Proof of Corollary 20. For simplicity, consider an alternative version of the sustainability constraint (from previous proof),

$$L_N^{cb}(\widetilde{i}_N) + \beta p L_Z^{cb}(\widetilde{i}_N) \geq L_N^{cb}\left(i_N^*\right) + \beta p L_Z^{cb}\left(i_N^*\right)$$

where I left arguments as functions of state-N nominal interest rates. Rearranging we obtain

$$H(\widetilde{i}_N) \equiv L_N^{cb}(\widetilde{i}_N) + \beta p L_Z^{cb}(\widetilde{i}_N) - L_N^{cb}\left(i_N^*\right) - \beta p L_Z^{cb}\left(i_N^*\right) \geq 0$$

The resulting $H(\widetilde{i}_N)$ function is continuous and concave, since the first two terms are (the negative of) two quadratic functions on $\widetilde{i}_N$, and the third and fourth terms are constants functions. Therefore, with standard arguments of continuous functions, if $H > 0$ at some $\widetilde{i}_N$, then there exists an $\widetilde{i}'_N$ in the neighborhood of $\widetilde{i}_N$ such that $H > 0$. ■

Proof (Asymptote of i_R^*). From Proposition 4, recall that $i_R^* = \theta_R^i r_Z + r_R$. Then,

$$\theta_R^i \equiv (\alpha_y^{CB}((1 - \left(\beta^{PS}\right)^2 p) + \kappa) + \alpha_\pi^{CB}\kappa^2(1 + \kappa + \beta^{PS}))p\upsilon/\phi$$

where

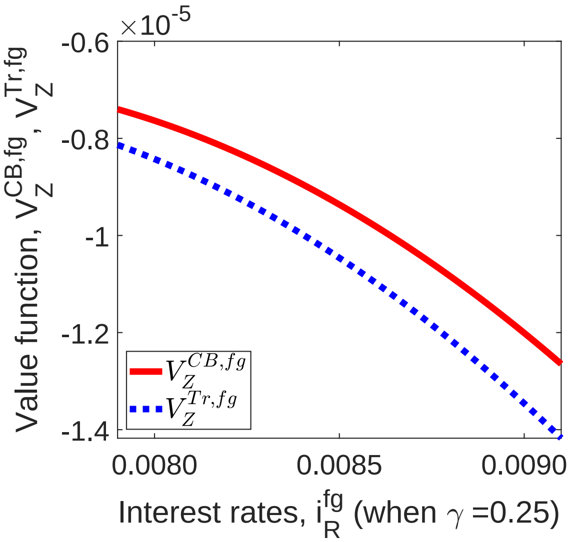
$$\phi \equiv \alpha_y^{CB}((1 - p\left(\beta^{PS}\right)^2)\gamma^2\alpha_y^{Tr} + (1 - p\beta^{PS}(\beta^{PS} + \kappa))\upsilon) + \alpha_\pi^{CB}\kappa^2(\gamma^2\alpha_y^{Tr} + (1 + \beta^{PS}p)\upsilon)$$

and $\upsilon \equiv \alpha_\tau^{Tr}\beta^{Tr}(1 + r_R)^2$. Note that ϕ can be rewritten as

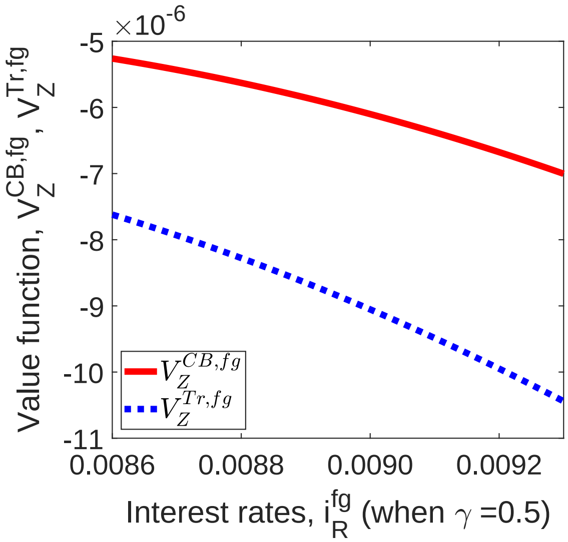
$$\phi \equiv \left[\alpha_y^{CB}\left(1 - p\left(\beta^{PS}\right)^2\right) + \alpha_\pi^{CB}\kappa^2\right]\gamma^2\alpha_y^{Tr} + H$$

with H terms independent from γ. Then, $\lim_{\gamma \to \infty} \phi \to \infty$, and variable $i_R^* = \theta_R^i r_Z + r_R$ tends to r_R. ■

Figure 9 **(Welfare)**



(A) Welfare ($\gamma = 0.25$).



(B) Welfare ($\gamma = 0.50$).

Figure 9: Welfare for different levels of γ.